An introduction to the symplectic embedding capacity of 4-dimensional ellipsoids into polydics

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Abstract

McDuff and Schlenk determined exactly when a four-dimensional symplectic ellipsoid symplectically embeds into a symplectic ball. Similarly, Frenkel and Müller determined exactly when a symplectic ellipsoid sympletically embeds into a symplectic cube. Symplectic embeddings of more complicated structures, however, remain mostly unexplored. Recently, Timmons, Burkhart and Panescu proved novel theorems concerning when a symplectic ellipsoid $E(a, b)$ symplectically embeds into a polydisc $P(c, d)$; this thesis is a survey of these results. We prove that there exists a constant depending only on d/c (here, d is assumed greater than c) such that if b/a is greater than said constant, then the only obstruction to symplectically embedding $E(a, b)$ into $P(c, d)$ is the volume obstruction. We also completely determine the symplectic embedding of $E(a, \frac{13}{2})$ into $P(c, d)$. Finally, we conjecture exactly when an ellipsoid embeds into a scaling of $P(1, b)$ for b greater than or equal to 6, and conjecture about the set of (a, b) such that the only obstruction to embedding $E(1, a)$ into a scaling of $P(1, b)$ is the classical volume.

Contents

1 Introduction

1.1 Background

This thesis includes a background and key sections of *Symplectic embeddings of 4-dimensional ellipsoids into polydiscs*, [13], which I co-wrote with Max Timmons and Madeleine Burkhart. This work was a result of our research at the UC Berkeley Geometry, Topology and Operators Algebra Research Training Group of Summer 2013. Here, a richer background on symplectic embeddings is offered, whereas [13] includes in-depth proofs of some of the more complicated theorems.

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1.2 Introduction to Symplectic Embeddings

To understand symplectic embeddings, it is important to note that although symplectic manifolds locally look like the standard structure on Euclidean space, the geometry is different from Riemannian geometry. The study of the rigidity of symplectic embeddings was popularized by Gromov in 1985. Gromov showed that volume preservation is necessary but insufficient for symplectic embeddings:

Theorem 1.1. *(Gromov's nonsqueezing Theorem) There exists a symplectic embedding of the ball $B(a)$ into the cylinder $Z(A)$ if and only if $a \leq A$.*

The volume of the cylinder $Z(A)$ is infinite, so it is clear that symplectic embeddings are more rigid than volume preserving embeddings. Work by

Gromov [6], McDuff-Polterovich [10] and Biran [2] suggests that this symplectic rigidity disappears for some values of a, although the precise barrier between symplectic and volume preserving embeddings is unknown.

Suppose X is a $2n$-dimensional smooth manifold and ω is a 2-form that is closed and nondegenerate. We say (X, ω) is a *symplectic* $2n$*-manifold* and call ω a *symplectic form* on X. If (X_0, ω_0) and (X_1, ω_1) are symplectic $2n$-manifolds then a *symplectic embedding* of (X_0, ω_0) into (X_1, ω_1), denoted $(X_0, \omega_0) \overset{s}{\hookrightarrow} (X_1, \omega_1)$, is a smooth embedding $\varphi : M_0 \to M_1$ such that $\varphi^*\omega_1 = \omega_0$. As discussed above, symplectic embeddings are volume preserving, thus it is necessary that $\mathrm{Vol}(X_0) \leq \mathrm{Vol}(X_1)$ when $(X_0, \omega_0) \overset{s}{\hookrightarrow} (X_1, \omega_1)$.

It is interesting to ask when one symplectic manifold embeds into another. For example, define the (open) four-dimensional symplectic *ellipsoid*

$$E(a, b) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} < 1 \right\}, \tag{1.1}$$

and define the (open) *symplectic ball* $B(a) := E(a, a)$. These inherit symplectic forms by restricting the standard form $\omega = \sum_{k=1}^{2} dx_k dy_k$ on $\mathbb{R}^4 = \mathbb{C}^2$. In [11], McDuff and Schlenk determined exactly when a four-dimensional symplectic ellipsoid $E(a, b)$ embeds symplectically into a symplectic ball, and found that if $\frac{b}{a}$ is small, then the answer involves an "infinite staircase" determined by the odd index Fibonacci numbers, while if $\frac{b}{a}$ is large then all obstructions vanish except for the volume obstruction.

Additionally, define the (open) four-dimensional *polydisc*

$$P(a, b) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \pi|z_1|^2 < a,\ \pi|z_2|^2 < b \right\}, \tag{1.2}$$

where $a, b \geq 1$ are real numbers and the symplectic form is again given by restricting the standard symplectic form on $\mathbb{R}^4$. Frenkel and Müller determined in [5] exactly when a four-dimensional symplectic ellipsoid symplectically embeds into a *cube* $C(a) := P(a, a)$ and found that part of the expression involves the Pell numbers. Cristofaro-Gardiner and Kleinman [4] studied embeddings of four-dimensional ellipsoids into scalings of $E(1, \frac{3}{2})$ and also found that part of the answer involves an infinite staircase determined by a recursive sequence.

Here we study symplectic embeddings of an open four-dimensional symplectic ellipsoid $E(a, b)$ into an open four-dimensional symplectic polydisc $P(c, d)$. By scaling, we can encode this embedding question as the function

$$d(a, b) := \inf\{\lambda | E(1, a) \overset{s}{\hookrightarrow} P(\lambda, b\lambda)\}, \tag{1.3}$$

where a and b are real numbers that are both greater than or equal to 1.

The function $d(a, b)$ always has a lower bound, $\sqrt{\dfrac{a}{2b}}$, equal to the volume obstruction.

1.3 Introduction to ECH Capacities

Any symplectic four-manifold has an associated sequence of nonnegative (possibly infinite) real numbers (indexed starting at 0) called *the embedded contact homology (ECH) capacities*, a concept developed by Michael Hutchings [7]. These ECH capacities obstruct symplectic embeddings. We only briefly discuss ECH capacities here; see [7] for a full survey.

The ECH capacities of a four-dimensional ellipsoid are equal to the sequence $N(a, b)$ of all non-negative integer linear combinations of a and b, arranged with repetitions in non-decreasing order. For example,

$$N(1, 1) = (0, 1, 1, 2, 2, 2, 3, 3, 3, 3, ...).$$

The ECH capacities of a four-dimensional polydisc are equal to $M(a, b)$, the sequence whose k^{th} term is

$$\min\{ma + nb|(m + 1)(n + 1) \geq k + 1\}$$

where $k, m, n \in \mathbb{Z}_{\geq 0}$. For example,

$$M(1, 1) = (0, 1, 2, 2, 3, 3, 4, 4, 4, 5, ...).$$

Write $N(a, b) \leq M(c, d)$ if each term in the sequence $N(a, b)$ is less than or equal to the corresponding term in $M(c, d)$. Frenkel and Müller show that embeddings of an ellipsoid into a polydisc are completely determined by the sequences M and N:

Theorem 1.2. *(Frenkel-Müller [5]) There is a symplectic embedding* $E(a, b) \overset{s}{\hookrightarrow} P(c, d)$ *if and only if* $N(a, b) \leq M(c, d)$.

Theorem 1.2 is equivalent to the statement that the ECH capacities give sharp obstructions to embeddings of an ellipsoid into a polydisc.

This theorem implies that we can rewrite (1.3) as

$$d(a, b) = sup\left\{\frac{N_k(1, a)}{M_k(1, b)} : k \in \mathbb{N}\right\}, \tag{1.4}$$

which helps with the construction and analysis of the graph of $d(a, b)$.

1.4 Statement of Results

Our first theorem states that for fixed b, if a is sufficiently large then all embedding obstructions vanish aside from the volume obstruction:

Theorem 1.3. *If* $a \geq \dfrac{9(b + 1)^2}{2b}$, *then* $d(a, b) = \sqrt{\dfrac{a}{2b}}$.

This is an analogue of a result of Buse-Hind [3] concerning symplectic embeddings of one symplectic ellipsoid into another.

From the previously mentioned work of McDuff-Schlenk, Frenkel-Müller, and Cristofaro-Gardiner-Kleinman, one expects that if a is small then the function $d(a,b)$ should be more rich. Our results suggest that this is indeed the case. For example, we completely determine the graph of $d(a, \frac{13}{2})$ (see Figure 1):

Theorem 1.4. *For* $b = \frac{13}{2}$, $d(a,b) \geq \sqrt{\frac{a}{13}}$, *i.e. the volume obstruction, and is equal to this lower bound for all* a *except on the following intervals:*

(i) $d(a, \frac{13}{2}) = 1$ *for all* $a \in \left[1, \frac{25}{2}\right]$

(ii) For $0 \leq k \leq 4$, $k \in \mathbb{Z}$:

$$d(a,b) = \begin{cases} \dfrac{2a}{25+2k} & a \in [\alpha_k, 13+2k], \\ \\ \dfrac{26+4k}{25+2k} & a \in [13+2k, \beta_k], \end{cases}$$

where $\alpha_0 = 25/2$, $\alpha_1 = 351/25$, $\alpha_2 = 841/52$, $\alpha_3 = 961/52$, $\alpha_4 = 1089/52$, $\beta_0 = 351/25$, $\beta_1 = 1300/81$, $\beta_2 = 15028/841$, $\beta_3 = 18772/961$, *and* $\beta_4 = 2548/121$.

Interestingly, the graph of $d(a, \frac{13}{2})$ has only finitely many nonsmooth points, in contrast to the infinite staircases in [11, 5, 4]. This is exactly why $b = \frac{13}{2}$, determining finitely many steps is a more approachable problem than determining infinitely many, and appears to be the case for many values of b. For example, we conjecture what the function $d(a,b)$ is for all $b \geq 6$, see conjecture 5.3.

2 Proof of Theorem 1.3

2.1 Weight sequences and the # operation

We begin by describing the machinery that will be used to prove Theorem 1.3.

Let a^2 be a nonnegative rational number. In [9], McDuff shows that there is a finite sequence of numbers

$$W(1,a^2) = (a_1, ..., a_n),$$

called the *(normalized) weight sequence for* a^2, such that $E(1,a^2)$ embeds into a symplectic ellipsoid if and only if the disjoint union $\sqcup B(W) := \sqcup B(a_i)$ embeds into that ellipsoid.

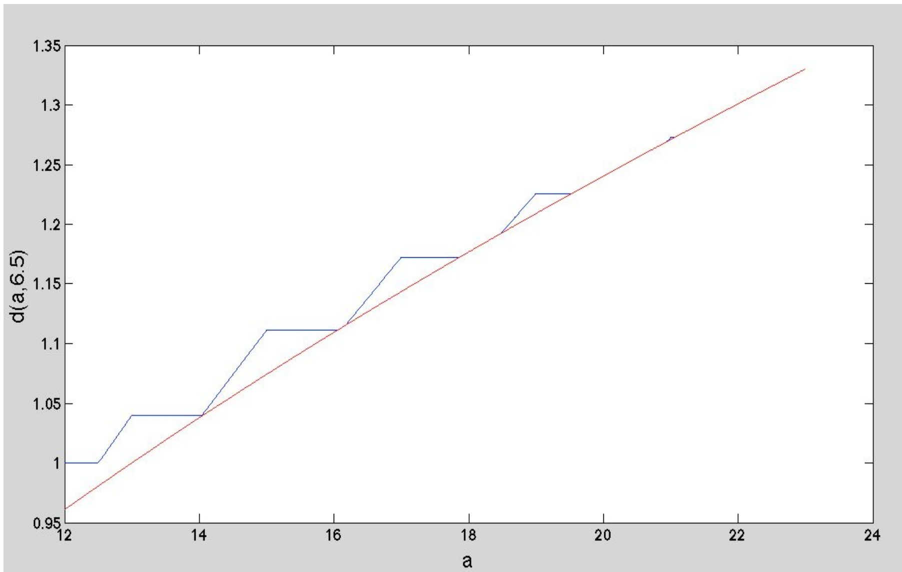

Figure 1: The graph of $d(a, \frac{13}{2})$. The red line represents the volume obstruction.

To describe the weight sequence, let

$$W(a^2, 1) = (X_0^{\times \ell_0}, X_1^{\times \ell_1}, ..., X_k^{\times \ell_k}) \tag{2.1}$$

where $X_0 > X_1 > ... > X_k$ and $\ell_k \geq 2$. The ℓ_i are the multiplicities of the entries X_i and come from the continued fraction expansion

$$a^2 = \ell_0 + \cfrac{1}{\ell_1 + \cfrac{1}{\ell_2 + ... \cfrac{1}{\ell_k}}} := [\ell_0; \ell_1, ..., \ell_k].$$

The entries of 2.1 are defined as follows:

$$X_{-1} := a^2, X_0 = 1, X_{i+1} = X_{i-1} - \ell_i X_i, i \geq 0.$$

Important properties of the weight sequence include that

$$\Sigma_i a_i^2 = a^2 \tag{2.2}$$

and

$$\Sigma_i a_i = a + 1 - \frac{1}{q} \tag{2.3}$$

where for all i, $a_i \leq 1$ and $a = \dfrac{p}{q}$.

We will also make use of a helpful operation, #, as in [9]. Suppose s_1 and s_2 are sequences indexed with $k \in \mathbb{Z}$, starting at 0. Then,

$$(s_1 \# s_2)_k = sup_{i+j=k}(s_1)_i + (s_2)_j.$$

A useful application of # is the following lemma:

Lemma 2.1. *(McDuff [9]) For all $a, b > 0$, we have $N(a,a)\#N(a,b) = N(a, a+b)$. More generally, for all $\ell \geq 1$, we have $(\#^\ell N(a,a))\#N(a,b) = N(a, b+\ell a)$.*

Note that the operation $\#^\ell$ is equivalent to applying # ℓ times. This lemma together with the weight sequence and scaling implies that

$$N(1, a^2) = N(a_1, a_1)\#...\#N(a_n, a_n). \tag{2.4}$$

Similar to McDuff [9], this machinery allows us to reduce Theorem 1.3 to a ball-packing problem.

2.2 Proof of Theorem 1.3

We begin by noting that the ECH capacities for $B(a)$ are

$$N(a,a) = (0, a, a, 2a, 2a, 2a, 3a, 3a, 3a, ...).$$

where the terms $N_k(a,a)$ of this sequence are of the form na and for each n there are $n + l$ entries occurring at

$$\frac{1}{2}(n^2 + n) \leq k \leq \frac{1}{2}(n^2 + 3n). \tag{2.5}$$

Similarly, for the sequence $\frac{a}{\sqrt{2b}}M(1,b)$, each term $\frac{a}{\sqrt{2b}}M_k(1,b)$ is of the form $n\frac{a}{\sqrt{2b}}$ where

$$k \leq \frac{n^2}{4b} + \frac{(1+b)n}{2b} + \frac{b^2 - 2b + 1}{4b}. \tag{2.6}$$

By scaling and continuity, we can study $d(a^2, b)$ with a^2 rational. So, we can prove that the volume obstruction is the only obstruction when $a \geq \frac{3(b+1)}{\sqrt{2b}}$ by showing that

$$N(1, a^2) \leq \frac{a}{\sqrt{2b}}M(1,b) \tag{2.7}$$

for said a values.

By 2.5 and 2.6, it is sufficient to show that

$$\Sigma_i n_i a_i \leq \frac{a}{\sqrt{2b}}n \tag{2.8}$$

whenever $n_1, .., n_m, n$ are nonnegative integers such that

$$\Sigma_i(n_i^2 + n_i) \leq 2(\frac{n^2}{4b} + \frac{(1+b)n}{2b} + \frac{b^2-2b+1}{4b}). \tag{2.9}$$

We do so by considering the following cases:

Case 1. $\Sigma_i(n_i^2) \leq \frac{n^2}{2b}$. In this case, the Cauchy-Schwartz Inequality along with 2.2 implies 2.8.

Case 2. $\Sigma_i(n_i^2) > \frac{n^2}{2b}$. This case along with 2.9 implies

$$\Sigma_i n_i a_i \leq \Sigma_i n_i \leq \frac{(1+b)n}{b} + \frac{b^2-2b+1}{2b}.$$

So, we need

$$\frac{(1+b)n}{b} + \frac{b^2-2b+1}{2b} \leq \frac{a}{\sqrt{2b}}n.$$

It follows that

$$a \geq \frac{b+1}{\sqrt{2b}}(2 + \frac{b+1}{n}). \tag{2.10}$$

Now let $n = b+1$. We see that in this case 2.6 is equivalent to

$$k \leq b + 1 + \frac{1}{4b}.$$

where $N_k(1, a^2) \leq \frac{a}{\sqrt{2b}} M_k(1, b)$ for all such k values. As such, we can set $n = b+1$ to 2.10 to get

$$a \geq \frac{3(b+1)}{\sqrt{2b}}, \tag{2.11}$$

hence the desired result. □

Remark 2.2. We allow $n = b+1$ in the statement of Theorem 1.4. However, if we show $N_k(1, a^2) \leq \frac{a}{\sqrt{2b}} M_k(1, b)$ for all $k \leq \frac{d^2}{4b} + \frac{(1+b)d}{2b} + \frac{b^2-2b+1}{4b}$ for some other value of n, then we can use this n in 2.10 to achieve a sharper bound for a.

3 Proof of Theorem 1.4 Part I

This section proves parts *(i)* and *(ii)* of Theorem 1.4. Specifically, we prove the exact intervals in which the graph of $d(a, \frac{13}{2})$ is linear. We begin by identifying the nondifferentiable points of $d(a, \frac{13}{2})$. These points were estimated using the graphing features of Matlab and then confirmed by using the Matlab code shown in the appendix. Here, we use Ehrhart polynomial theory to prove the existence of these nondifferentiable points. Using monotonicity and subscaling arguments (see Lemmas 3.3 and 3.4), we prove the existence of the linear steps of the graph of d.

3.1 Nondifferentiable points and Ehrhart polynomials

We first compute the values of d at certain points. These will eventually be the points a where $d(a, \frac{13}{2})$ is not differentiable.

Proposition 3.1. *We have:*

$$\begin{aligned}
&d\left(1, \frac{13}{2}\right) = 1, && d\left(\frac{25}{2}, \frac{13}{2}\right) = 1, && d\left(13, \frac{13}{2}\right) = \frac{26}{25}, \\
&d\left(\frac{351}{25}, \frac{13}{2}\right) = \frac{26}{25}, && d\left(15, \frac{13}{2}\right) = \frac{10}{9}, && d\left(\frac{1300}{81}, \frac{13}{2}\right) = \frac{10}{9}, \\
&d\left(\frac{841}{52}, \frac{13}{2}\right) = \frac{29}{26}, && d\left(17, \frac{13}{2}\right) = \frac{34}{29}, && d\left(\frac{15028}{841}\right) = \frac{34}{29}, \\
&d\left(\frac{961}{52}, \frac{13}{2}\right) = \frac{31}{26}, && d\left(19, \frac{13}{2}\right) = \frac{38}{31}, && d\left(\frac{18772}{961}, \frac{13}{2}\right) = \frac{38}{31}, \\
&d\left(\frac{1089}{52}, \frac{13}{2}\right) = \frac{33}{26}, && d\left(21, \frac{13}{2}\right) = \frac{42}{33}, \text{ and } && d\left(\frac{2548}{121}, \frac{13}{2}\right) = \frac{42}{33}.
\end{aligned}$$

To prove the proposition, the main difficulty comes from the fact that that applying Theorem 1.2 in principle requires checking infinitely many ECH capacities. Our strategy for overcoming this difficulty is to study the growth rate of the terms in the sequences M and N. We will find that in every case needed to prove Proposition 3.1, one can bound these growth rates to conclude that only finitely many terms in the sequences need to be checked. This is then easily done by computer. The details are as follows:

Proof. Step 1. For the sequence $N(a, b)$, let $k(a, b, t)$ be the largest k such that $N_k(a, b) \leq t$. Similarly, for the sequence $M(c, d)$, let $l(c, d, t)$ be the largest l such that $M_l(c, d) \leq t$. To show that $E(a, b) \overset{s}{\hookrightarrow} P(c, d)$, by Theorem 1.2, we just have to show that for all t, we have $k(a, b, t) \geq l(c, d, t)$.

Step 2. We can estimate $k(a, b, t)$ by applying the following proposition:

Proposition 3.2. *If $a, b, r,$ and t are all positive integers, then* $k(\frac{a}{r}, \frac{b}{r}, t) =$

$$\frac{1}{2ab}(tr)^2 + \frac{1}{2}(tr)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{ab}\right) + \frac{1}{4}\left(1 + \frac{1}{a} + \frac{1}{b}\right) + \frac{1}{12}\left(\frac{a}{b} + \frac{b}{a} + \frac{1}{ab}\right) + \frac{1}{a}\sum_{j=1}^{a-1} \frac{\xi_a^{j(-tr)}}{(1 - \xi_a^{jb})(1 - \xi_a^j)} + \frac{1}{b}\sum_{l=1}^{b-1} \frac{\xi_b^{l(-tr)}}{(1 - \xi_b^{la})(1 - \xi_b^l)}, \tag{3.1}$$

where $\xi_d = e^{\frac{2\pi i}{d}}$.

Proof. The number of terms in $N(\frac{a}{r}, \frac{b}{r})$ that are less than t is the same as the number of lattice points (m, n) in the triangle bounded by the positive x and y axes and the line $x\frac{a}{r} + y\frac{b}{r} \leq t$. For integral t, this number can be computed by applying the theory of "Ehrhart polynomials". Proposition 3.2 follows by applying [1, Thm. 2.10]. □

Motivated by the definition of d, we will be most interested in this proposition in the case where $a = r$. Note that by the last two terms of the formula in Proposition 3.2, $k(\frac{a}{r}, \frac{b}{r}, t)$ is a periodic polynomial with period ab.

We also need an argument to account for the fact that Proposition 3.2 is only for integer t, whereas the argument in step 1 involves real t. To account for this, we use an asymptotic argument. Specifically, for $E(1, \frac{a}{r})$, $a, r \in \mathbb{Z}_{\geq 1}$, we bound the right hand side of (3.1) from below by taking the floor function of t. It is convenient for our argument to further bound this expression from below by

$$\frac{c_1}{r^2}(rt-1)^2 + \frac{c_2}{r}(rt-1) + c_3. \tag{3.2}$$

where the c_i are the coefficients of the right hand side of (3.1) that do not involve t or r.

This is the lower bound that we will use for $k(1, \frac{a}{r}, t)$.

Step 3. To get an upper bound $l(c, d, t)$ for $M(c, d)$, recall that $M_l(c, d) = min\{cm + dn : (m + 1)(n + 1) \geq l + 1\}$. For $cm + dn = t$, we solve for m in terms of n and find:

$$\left(\frac{t-dn}{c} + 1\right)(n+1) - 1 \geq l.$$

Considering $m, n \in \mathbb{R}$, we can take the derivative of the left side of the inequality with respect to n and then set the expression equal to 0 to maximize it. We do the same with m to obtain:

$$\left(\frac{t}{2d} + \frac{c}{2d} + \frac{1}{2}\right)\left(\frac{t}{2c} + \frac{d}{2c} + \frac{1}{2}\right) - 1 \geq l.$$

Simplifying, we find that an upper bound for l is:

$$l(c, d, t) = \frac{t^2}{4cd} + \frac{(c+d)t}{2cd} + \frac{(c-d)^2}{4cd}. \tag{3.3}$$

Our strategy now is to check for each point in Proposition 3.1 that we have $k(a, b, t) \geq l(c, d, t)$ asymptotically in t for the corresponding (a, b, c, d). From there, we can check that for a sufficient number of terms, $N(1, a) \leq M(\lambda, \lambda b)$.

Step 4. Since the rest of the proof amounts to computation, it is best summarized by the chart below. In the chart, k_{t^2} and l_{t^2} denote the coefficients of the quadratic terms in the upper and lower bounds from steps 2 and 3, while k_t and l_t denote the corresponding coefficients of the linear terms.

The t column gives the sufficient number of t terms to check up to before the asymptotic bounds from the previous three steps are enough. Note that if the t^2 coefficients in any row are equal, then the linear coefficients are used to make an asymptotic argument; this explains the appearance of the "N/A"s in the table. It is simple to check by computer that the relevant N and M sequences in each row satisfy $N \leq M$ once one knows that the problem only has to be checked up to the term in the t column. Code for this is contained in A.

The ECH obstruction column gives an ECH capacity that shows that one cannot shrink λ further, i.e. the claimed embeddings are actually sharp.

$E(1,a) \overset{s}{\hookrightarrow} P(\lambda, \lambda b)$	k_{t^2}	l_{t^2}	k_t	l_t	t	ECH obstruction
$E(1, \frac{25}{2}) \overset{s}{\hookrightarrow} P(1, \frac{13}{2})$	$\frac{1}{25}$	$\frac{1}{26}$	N/A	N/A	51	1
$E(1, 13) \overset{s}{\hookrightarrow} P(\frac{26}{25}, \frac{169}{25})$	$\frac{1}{26}$	$\frac{625}{17576}$	N/A	N/A	33	13
$E(1, \frac{351}{25}) \overset{s}{\hookrightarrow} P(\frac{26}{25}, \frac{169}{25})$	$\frac{25}{702}$	$\frac{625}{17576}$	N/A	N/A	522	13
$E(1, 15) \overset{s}{\hookrightarrow} P(\frac{10}{9}, \frac{65}{9})$	$\frac{1}{30}$	$\frac{81}{2600}$	N/A	N/A	29	15
$E(1, \frac{1300}{81}) \overset{s}{\hookrightarrow} P(\frac{10}{9}, \frac{65}{9})$	$\frac{81}{2600}$	$\frac{81}{2600}$	$\frac{691}{1300}$	$\frac{27}{52}$	272	15
$E(1, \frac{841}{52}) \overset{s}{\hookrightarrow} P(\frac{29}{26}, \frac{29}{4})$	$\frac{26}{841}$	$\frac{26}{841}$	$\frac{447}{841}$	$\frac{15}{29}$	122	17
$E(1, 17) \overset{s}{\hookrightarrow} P(\frac{34}{29}, \frac{221}{29})$	$\frac{1}{34}$	$\frac{841}{30056}$	N/A	N/A	27	17
$E(1, \frac{15028}{841}) \overset{s}{\hookrightarrow} P(\frac{34}{29}, \frac{221}{29})$	$\frac{841}{30056}$	$\frac{841}{30056}$	$\frac{7935}{15028}$	$\frac{435}{884}$	32	17
$E(1, \frac{961}{52}) \overset{s}{\hookrightarrow} P(\frac{31}{26}, \frac{31}{4})$	$\frac{26}{961}$	$\frac{26}{961}$	$\frac{507}{961}$	$\frac{15}{31}$	23	19
$E(1, 19) \overset{s}{\hookrightarrow} P(\frac{38}{31}, \frac{247}{31})$	$\frac{1}{38}$	$\frac{961}{37544}$	N/A	N/A	7	19
$E(1, \frac{18772}{961}) \overset{s}{\hookrightarrow} P(\frac{38}{31}, \frac{247}{31})$	$\frac{961}{37544}$	$\frac{961}{37544}$	$\frac{759}{1444}$	$\frac{465}{988}$	28	19
$E(1, \frac{1089}{52}) \overset{s}{\hookrightarrow} P(\frac{33}{26}, \frac{33}{4})$	$\frac{26}{1089}$	$\frac{26}{1089}$	$\frac{571}{1089}$	$\frac{15}{33}$	14	21
$E(1, 21) \overset{s}{\hookrightarrow} P(\frac{42}{33}, \frac{273}{33})$	$\frac{1}{42}$	$\frac{121}{5096}$	N/A	N/A	26	21
$E(1, \frac{2548}{121}) \overset{s}{\hookrightarrow} P(\frac{42}{33}, \frac{273}{33})$	$\frac{121}{5096}$	$\frac{121}{5096}$	$\frac{1335}{2548}$	$\frac{165}{364}$	41	21

Table 3.1

□

3.2 The linear steps

Given the computations from the previous section, the computation of $d(a, \frac{13}{2})$ for all the "linear steps", i.e. those portions of the graph of d for which d is linear, is straightforward. Indeed, we have the following two lemmas:

Lemma 3.3. *For a fixed b, $d(a,b)$ is monotonically non-decreasing.*

Proof. This follows from the fact that $E(1,a) \overset{s}{\hookrightarrow} E(1,a^{'})$ if $a \leq a^{'}$. $\square$

Lemma 3.4. $d(\lambda a, b) \leq \lambda d(a,b)$. *(subscaling)*

Proof. This follows from the fact that $E(1,\lambda a) \overset{s}{\hookrightarrow} E(\lambda, \lambda a)$ for $\lambda \geq 1$. $\square$

By monotonicity, we know that $d(a, \frac{13}{2})$ is constant on the intervals:

$$a \in \left[1, \frac{25}{2}\right], \left[13, \frac{351}{25}\right], \left[15, \frac{1300}{81}\right], \left[17, \frac{15028}{841}\right], \left[19, \frac{18772}{961}\right], \left[21, \frac{2548}{121}\right].$$

We now explain why for $0 \leq k \leq 4, k \in \mathbb{Z}$,

$$d(a, \tfrac{13}{2}) = \frac{2a}{25+2k} \quad a \in [\alpha_k, 13+2k],$$

where $\alpha_0 = \dfrac{25}{2}, \alpha_1 = \dfrac{351}{25}, \alpha_2 = \dfrac{841}{52}, \alpha_3 = \dfrac{961}{52}$, and $\alpha_4 = \dfrac{1089}{52}$.

Given the critical points we have determined, along with the subscaling lemma, we have $\dfrac{2a}{25+2k}$ as an upper bound for $d(a, \frac{13}{2})$ on the above intervals.

We also know that:

$$d(a, \tfrac{13}{2}) = sup\left\{\frac{N_x(1,a)}{M_x(1,\frac{13}{2})} : x \in \mathbb{N}\right\} \geq \frac{N_l(1,a)}{M_l(1,\frac{13}{2})} \text{ for any } l.$$

Here is a representative example of our method:

Example 3.5. To illustrate how this can give us a suitable lower bound, consider the case where $x = 13$:

$$sup\left\{\frac{N_x(1,a)}{M_x(1,\frac{13}{2})} : x \in \mathbb{N}\right\} \geq \frac{N_{13}(1,a)}{M_{13}(1,\frac{13}{2})} = \frac{2a}{25}$$

for $a \in \left[\frac{25}{2}, 13\right]$.

This lower bound equals the upper bound given by Lemma 3.4, so we have proven Theorem 1.4 for $a \in [\frac{25}{2}, 13]$.

The general method is similar: given $a \in [\alpha_k, 13+2k]$, we can find an l such that:

$$\frac{N_l(1,a)}{M_l(1,\frac{13}{2})} = \frac{2a}{25+2k}.$$

Such obstructing values of l are given in the following table:

k	$\dfrac{2}{25+2k}$	l
0	$\frac{2}{25}$	13
1	$\frac{2}{27}$	15
2	$\frac{2}{29}$	17
3	$\frac{2}{31}$	19
4	$\frac{2}{33}$	21

Table 3.2

Given $a \in [\alpha_k, 13+2k]$ for each integer $k \in [0,4]$, we have found that the upper and lower bounds of $d(a, \frac{13}{2})$ equal $\dfrac{2a}{25+2k}$. Thus, we have proven our claim for these intervals.

4 Proof of Theorem 1.4 Parts II and III

To finish the proof of Theorem 1.4, we must prove that all other, nonlinear segments of the graph of d are equal to the volume obstruction. The second part of the proof of Theorem 1.4 involves complicated machinery that will not be discussed here. The proof adapts ideas from [11] and [9] in a purely combinatorial way and involves the use of the Cremona transform and reducing the proof to a ball packing problem, as in the proof of Theorem 1.3. Please refer to [13] for the in-depth proofs that show that $d(a, \frac{13}{2})$ equals $\sqrt{\dfrac{a}{13}}$, i.e. the volume obstruction, for the following intervals of a: $[\frac{1300}{81}, \frac{841}{52}]$, $[\frac{15028}{841}, \frac{961}{52}]$, $[\frac{18772}{961}, \frac{1089}{52}]$, $[\frac{2548}{121}, 27]$, and $[27, \infty)$.

5 Conjectures

We now present some conjectures concerning exactly when an ellipsoid embeds into a polydisc.

5.1 Extensions of Theorem 1.3

To consider an interesting refinement of Theorem 1.3, define $V(b) = \inf\{A : d(a,b) = \sqrt{\frac{a}{2b}} \text{ for } a \geq A\}$. Theorem 1.3 implies $V(b) \leq \frac{9}{2}(b + 2 + \frac{1}{b})$.

Proposition 5.1. *For $b \geq 1$*

$$V(b) \geq 2b\left(\frac{2\lfloor b \rfloor + 2\left\lceil \sqrt{2b} + \{b\}\right\rceil - 1}{b + \lfloor b \rfloor + \left\lceil \sqrt{2b} + \{b\}\right\rceil - 1}\right)^2$$

where $\{b\} = b - \lfloor b \rfloor$.

Proof.

$$d(2\lfloor b\rfloor+2\left\lceil\sqrt{2b}+\{b\}\right\rceil-1,b) \geq \frac{N_{2\lfloor b\rfloor+2\lceil\sqrt{2b}+\{b\}\rceil-1}(1,2\lfloor b\rfloor+2\left\lceil\sqrt{2b}+\{b\}\right\rceil-1)}{M_{2\lfloor b\rfloor+2\lceil\sqrt{2b}+\{b\}\rceil-1}(1,b)}$$

$$= \frac{2\lfloor b\rfloor+2\left\lceil\sqrt{2b}+\{b\}\right\rceil-1}{b+\lfloor b\rfloor+\left\lceil\sqrt{2b}+\{b\}\right\rceil-1}$$

$$> \sqrt{\frac{2\lfloor b\rfloor+2\left\lceil\sqrt{2b}+\{b\}\right\rceil-1}{2b}}$$

This implies

$$V(b) \geq 2b\left(\frac{2\lfloor b\rfloor+2\left\lceil\sqrt{2b}+\{b\}\right\rceil-1}{b+\lfloor b\rfloor+\left\lceil\sqrt{2b}+\{b\}\right\rceil-1}\right)^2.$$

□

Experimental evidence seems to suggest that for $b > 1$ this bound is sharp.

Conjecture 5.2. *For* $b > 1$

$$V(b) = 2b\left(\frac{2\lfloor b\rfloor+2\left\lceil\sqrt{2b}+\{b\}\right\rceil-1}{b+\lfloor b\rfloor+\left\lceil\sqrt{2b}+\{b\}\right\rceil-1}\right)^2.$$

5.2 Generalizations of Theorem 1.2

The methods used to compute the graph of $d(a, 6.5)$ should extend for the most part to any b. In light of those techniques, experimental evidence, and a conjecture regarding $d(a, b)$ for b an integer by David Frenkel and Felix Schlenk relayed to us by Daniel Cristofaro-Gardiner, we offer a conjecture regarding the graph of $d(a, b)$ for $b \geq 6$, see Figure 2.

Conjecture 5.3. *For* $b \geq 6, d(a,b) = \sqrt{\frac{a}{2b}}$ *except on the following intervals:*
(i) $d(a,b) = 1$ *for all* $a \in [1, b+\lfloor b\rfloor]$
(ii) For $0 \leq k \leq \sqrt{2b}+\{b\}$, $k \in \mathbb{Z}$*:*

$$d(a,b) = \begin{cases} \dfrac{a}{b+\lfloor b\rfloor+k} & a \in [\alpha_k, 2(\lfloor b\rfloor+k)+1], \\ \\ \dfrac{2(\lfloor b\rfloor+k)+1}{b+\lfloor b\rfloor+k} & a \in [2(\lfloor b\rfloor+k)+1, \beta_k], \end{cases}$$

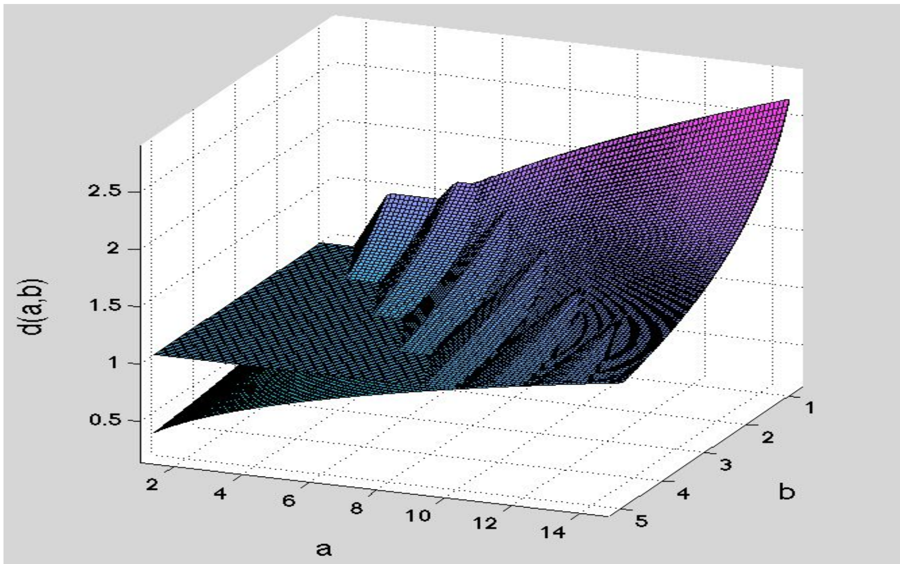

Figure 2: Approximate plot of the graph of $d(a,b)$

.

where $\alpha_0 = b + \lfloor b \rfloor, \alpha_1 = \beta_0 = \frac{(b+\lfloor b\rfloor+1)(2\lfloor b\rfloor+1)}{b+\lfloor b\rfloor}, \alpha_k = \frac{(b+\lfloor b\rfloor+k)^2}{2b}$ *for* $k \geq 2, and \beta_k = 2b\left(\frac{2(\lfloor b\rfloor+k)+1)}{b+\lfloor b\rfloor+k}\right)^2$ *for* $k \geq 1$.

For integers m*, if* $b \in [m - \frac{m}{(m+1)^2}, m + \frac{1}{2+m}]$ *then let* $b = m + \varepsilon$*. It follows that*

$$d(a,b) = \begin{cases} \dfrac{ma+1}{2m^2+(2+\varepsilon)m+\varepsilon} & a \in [\alpha^*, 2m+4], \\ \\ \dfrac{m(2m+4)+1}{2m^2+(2+\varepsilon)m+\varepsilon} & a \in [2m+4, \beta^*], \end{cases}$$

where $\alpha^* = \frac{1}{2(2m^3+2m^2\varepsilon)}(8m^3+4m^2+8m^2\varepsilon+4m^3\varepsilon+\varepsilon^2+2m\varepsilon^2+b^2\varepsilon^2-(1+m)(2m+\varepsilon)\sqrt{-4m^2+8m^3+4m^4-4m\varepsilon+8m^2\varepsilon+4m^3\varepsilon+\varepsilon^2+2m\varepsilon^2+m^2\varepsilon^2})$ *and* $\beta^* = \frac{2(\varepsilon+m+8m\varepsilon+8m^2+20m^2\varepsilon+16m^3\varepsilon+16m^4+4m^4\varepsilon+4m^5)}{(1+m)^2(2m+\varepsilon)^2}$.

We note that Conjecture 5.3 implies Conjecture 5.2 for $b \geq 6$. Furthermore, we prove that the conjecture is a lower bound for $d(a,b)$.

Proposition 5.4. *For* $b \geq 6, d(a,b) \geq \sqrt{\frac{a}{2b}}$ *and*
(i) $d(a,b) \geq 1$ *for all* $a \in [1, b + \lfloor b \rfloor]$
(ii) For $0 \leq k \leq \sqrt{2b} + \{b\}$, $k \in \mathbb{Z}$*:*

$$d(a,b) \geq \begin{cases} \dfrac{a}{b+\lfloor b\rfloor+k} & a \in [\alpha_k, 2(\lfloor b\rfloor+k)+1], \\ \\ \dfrac{2(\lfloor b\rfloor+k)+1}{b+\lfloor b\rfloor+k} & a \in [2(\lfloor b\rfloor+k)+1, \beta_k], \end{cases}$$

where $\alpha_0 = b + \lfloor b\rfloor, \alpha_1 = \beta_0 = \frac{(b+\lfloor b\rfloor+1)(2\lfloor b\rfloor+1)}{b+\lfloor b\rfloor}, \alpha_k = \frac{(b+\lfloor b\rfloor+k)^2}{2b}$ *for* $k \geq 2, \beta_k = 2b\left(\frac{2(\lfloor b\rfloor+k)+1)}{b+\lfloor b\rfloor+k}\right)^2$ *for* $k \geq 1$.

For integers m*, if* $b \in [m - \frac{m}{(m+1)^2}, m + \frac{1}{2+m}]$ *then let* $b = m + \varepsilon$*. It follows that*

$$d(a,b) \geq \begin{cases} \dfrac{ma+1}{2m^2+(2+\varepsilon)m+\varepsilon} & a \in [\alpha^*, 2m+4], \\ \\ \dfrac{m(2m+4)+1}{2m^2+(2+\varepsilon)m+\varepsilon} & a \in [2m+4, \beta^*], \end{cases}$$

where $\alpha^* = \frac{1}{2(2m^3+2m^2\varepsilon)}(8m^3+4m^2+8m^2\varepsilon+4m^3\varepsilon+\varepsilon^2+2m\varepsilon^2+b^2\varepsilon^2-(1+m)(2m+\varepsilon)\sqrt{-4m^2+8m^3+4m^4-4m\varepsilon+8m^2\varepsilon+4m^3\varepsilon+\varepsilon^2+2m\varepsilon^2+m^2\varepsilon^2})$ *and* $\beta^* = \frac{2(\varepsilon+m+8m\varepsilon+8m^2+20m^2\varepsilon+16m^3\varepsilon+16m^4+4m^4\varepsilon+4m^5)}{(1+m)^2(2m+\varepsilon)^2}$.

Proof. We know that $d(a,b) \geq \sqrt{\frac{a}{2b}}$ because symplectic embeddings are volume preserving. We also have

$$d(a,b) \geq \frac{N_1(1,a)}{M_1(a,b)} = \frac{1}{1} = 1.$$

Additionally, for $k \in \mathbb{Z}, 0 \leq k < \sqrt{2b} + \{b\}, a \in [2(\lfloor b\rfloor+k), 2(\lfloor b\rfloor+k)+1]$

$$d(a,b) \geq \frac{N_{2(\lfloor b\rfloor+k)+1}(1,a)}{M_{2(\lfloor b\rfloor+k)+1}(1,b)} = \frac{a}{b+\lfloor b\rfloor+k}$$

$$\geq 1 \text{ for } a \in [b+\lfloor b\rfloor, 2\lfloor b\rfloor+1], k=0$$

$$\geq \frac{2\lfloor b\rfloor+1}{b+\lfloor b\rfloor} \text{ for } a \in [\frac{(b+\lfloor b\rfloor+1)(2\lfloor b\rfloor+1)}{b+\lfloor b\rfloor}, 2\lfloor b\rfloor+3], k=1$$

$$\geq \sqrt{\frac{a}{2b}} \text{ for } a \in [\alpha_k, 2(\lfloor b\rfloor+k)+1], k \geq 2.$$

We also have for $a \in [2(\lfloor b\rfloor+k)+1, \infty)$

$$d(a,b) \geq \frac{N_{2(\lfloor b\rfloor+k)+1}(1,a)}{M_{2(\lfloor b\rfloor+k)+1}(1,b)} = \frac{2(\lfloor b\rfloor+k)+1}{b+\lfloor b\rfloor+k}$$

$$\geq \sqrt{\frac{a}{2b}} \text{ for } a \in [2(\lfloor b \rfloor + k) + 1, \beta_k].$$

Furthermore, if $b \in [m - \frac{m}{(m+1)^2}, m + \frac{1}{2+m}]$ for some $m \in \mathbb{Z}$ and $a \in [2m + 4 - \frac{1}{m}, 2m + 4]$

$$d(a,b) \geq \frac{N_{(m+1)^3}(1,a)}{M_{(m+1)^3}(1,b)} = \frac{ma+1}{2m^2 + (2+\varepsilon)m + \varepsilon}$$

$$\geq \sqrt{\frac{a}{2b}} \text{ for } a \in [\alpha^*, 2m+4].$$

We also have for $a \in [2m + 4m\beta^*]$

$$d(a,b) \geq \frac{N_{(m+1)^3}(1,a)}{M_{(m+1)^3}(1,b)} = \frac{m(2m+4)+1}{2m^2 + (2+\varepsilon)m + \varepsilon}$$

$$\geq \sqrt{\frac{a}{2b}} \text{ for } a \in [2m+4, \beta^*].$$

This completes the proof. □

A Appendix: Code that checks through terms of N and M

The following is Matlab code that allows us to check whether $N(1,b) \leq M(c,d)$ up through $N(1,a) \leq x$ (note that for the function *Membed*, $d \leq c$):

```
function m=embed(b,c,d,x)
l=length(Nembed(x,b));
y=zeros(1,l); w=Nembed(x,b); t=Membed(l,c,d);
for i=1:l
    if w(i)<=(t(i)+10^-10)
        y(i)=1;
    else
        m=0;
    break
    end
end
m=min(y);

function y=Nembed(x,b);
y=zeros(1,x+2);
for i=1:x+2
    y(i)=floor((x+b-(i-1))/b);
end
M=sum(y);
z=zeros(1,(x+1)^2);
for i=1:x+1
    for j=1:x+1
        z(i+(x+1)*(j-1))=i-1+(j-1)*b;
    end
end
l=sort(z);
y=zeros(1,M);
for i=1:M
    y(i)=l(i);
end

function q=Membed(N,c,d)
q=zeros(1,N);
for k=0:N-1
    w=zeros(1,ceil(sqrt(k+1)-1));
    for i=0:(ceil(sqrt(k+1)-1))
        w(i+1)=c*i+d*(ceil((k+1)/(i+1))-1);
```

```
    end
    q(k+1)=min(w);
end
```

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