Energy Minimization of Knotted Vortex Tubes

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1 Introduction

The vorticity field of an incompressible, homogenous fluid is frozen, meaning that the topological structure of the field is preserved by fluid motion. Disregarding the possibility of singularities or discontinuities and assuming the absence of external forces, we may use topological invariants as fluid flow invariants. The goal of this paper is to analyze the effects of topological structure on the energy dissipation of a fluid flow field. We will show that for a nontrivial topology, the energy of a vorticity field tends to a positive minimum which can be estimated through the application of topological invariants.

We consider the total energy of a viscous fluid in two components: the energy of the velocity field, called kinetic energy, and the energy of the vorticity field. We first prove that the energy of the vorticity field is monotonically decreasing as it is converted to kinetic energy, and then that the total energy is monotonically decreasing as kinetic energy is lost due to viscosity (Moffatt, [11]).

In section 3 we introduce several knot invariants, particularly the Gauss and Călugăreanu-White linking numbers, and discuss their relationship to the helicity of a vortex tube, a known invariant of fluid motion which provides a lower bound for the energy of the vorticity field (Moffatt, [10]). Calculation of the Gauss and Călugăreanu-White linking numbers depends upon the assignment of a value which may be either positive or negative to each link or knot crossing. The sum of a positive and negative crossing will then cancel out, sometimes yielding an overall sum of zero for a nontrivial structure. Thus the helicity of a nontrivial vortex tube may be zero, and so helicity is insufficient as a lower bound. In the following section, we refine the lower bound by instead using asymptotic crossing number as our invariant. Unlike the Gauss and Călugăreanu-White linking numbers, the asymptotic crossing number takes the absolute value at each crossing and hence sums to a positive value for every nontrivial knot (Freedman & He, [5]). We will show that the asymptotic crossing number also provides a lower bound for the energy of the vorticity field, which therefore tends to a positive limit because it is both monotonically decreasing and bounded below.

Having proved the existence of a lower limit, we attempt to attain the energy minimum by adopting a non-orthogonal coordinate system. Transformation and manipulation of the energy integral of a knotted, uniform vortex tube of circular cross-section results in an energy equation based purely on tube length and several fluid and topological invariants. If

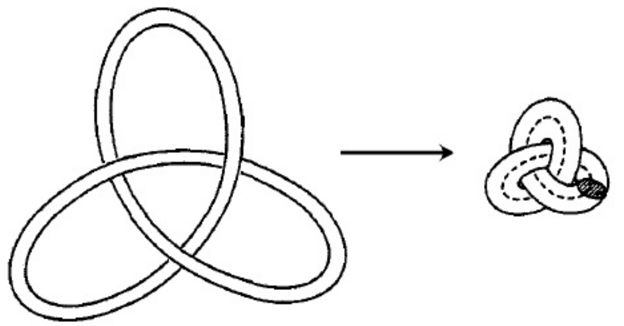

Figure 1: Deformation of a knotted vortex tube. The energy of a trefoil knot of uniform circular cross-section is minimized when $L \approx V^{1/3}$ and $A \approx V^{2/3}$. Source: Moffatt [10].

a knotted vortex tube does not satisfy the additional constraints of uniformity and circular cross-section, the equation provides an upper bound for the energy. We conclude that for a typical knotted vortex tube, tube length decreases as energy dissipates. Because volume is preserved, the cross-sectional area must increase to compensate for loss of length, as depicted by the trefoil knot in Figure 1. However, conservation of topology means that the tube cannot pass through itself, and it is for this reason that the lower bound attained in the early sections of the paper exists. Furthermore, in some cases a vortex tube may be so strongly knotted that the energy is minimized with an increase in length and the associated decrease in cross-sectional area (Chui & Moffatt, [2]).

In addition to the constraints of incompressibility and homogeneity, we presume a smooth velocity field. In the absence of smoothness, discontinuities may form which allow for alterations to the vortex topology. For a more detailed explanation of these discontinuities, see Moffatt [10].

2 The Movement of Fluid Energies

We consider an incompressible fluid of uniform density ρ with smooth velocity field $\mathbf{v}$.

Definition 2.1. *The vorticity $\boldsymbol{\omega}$ of a fluid flow field is the curl of the velocity:*

$$\boldsymbol{\omega} = \nabla \times \mathbf{v}.$$

Vorticity is a function of position $\mathbf{x}$ and time t, so we write $\boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{x}, t)$.

The vorticity vector at a point $\mathbf{x}$ is roughly twice the mean angular velocity vector at $\mathbf{x}$ with respect to the center of mass of a small neighborhood of particles, and hence is a measure of local "spin", oriented according to the right-hand rule. A fluid is said to be irrotational where the vorticity is 0. A vortex line is a curve that is everywhere tangent to the vorticity vector, and a vortex tube is a region of isolated vorticity; more precisely, it is the set of vortex lines passing through a closed curve C outside of which the vorticity is locally 0.

Definition 2.2. *The circulation (strength) Φ of a vortex tube is the line integral of the velocity field around the boundary curve C:*

$$\Phi = \int_C \mathbf{v} \cdot d\mathbf{l}.$$

Equivalently (by Stokes' theorem), the circulation is the flux of the vorticity over the cross-sectional surface S bounded by C:

$$\Phi = \iint_S \boldsymbol{\omega} \cdot d\mathbf{S}.$$

Unlike vorticity, circulation is a uniform scalar throughout a vortex tube, which means it can be calculated using any cross-sectional surface.

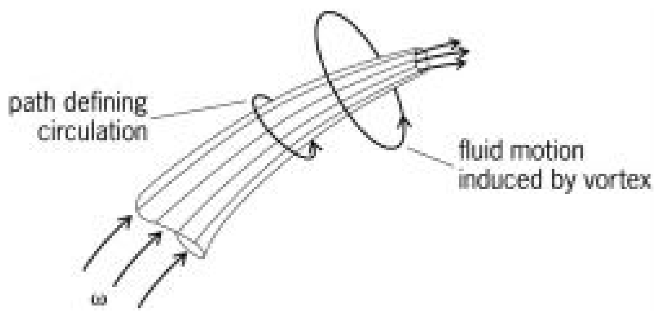

Figure 2: The relative orientations of circulation and vorticity are determined by the right-hand rule. Source: thefreedictionary.com

Viscous fluid motion is governed by the Navier-Stokes equations. For incompressible, homogenous fluids that are not acted upon by external forces, the equations that express conservation of matter and of momentum reduce to, respectively,

$$\nabla \cdot \mathbf{v} = 0 \tag{1}$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mathbf{F} + \mu \nabla^2 \mathbf{v} \tag{2}$$

where p is a pressure scalar, μ is the viscosity (friction between fluid particles), and $\mathbf{F} = (\nabla \times \boldsymbol{\omega}) \times \boldsymbol{\omega}$ is the force exerted by the vorticity field.

With these conditions, circulation is conserved. Not only is it uniform for a given vortex tube; it is constant over time. Furthermore, because vorticity is solenoidal ($\nabla \cdot \boldsymbol{\omega} = 0$), it is frozen under the Navier-Stokes equations, meaning that it satisfies the frozen field equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) \tag{3}$$

and the volume and field topology are preserved by the fluid flow; however, within the constraints of that topology certain deformations may occur. Stretching of a vortex tube, when the area of every cross-sectional surface decreases while the tube length increases, is typically associated with an energy cascade that causes the magnitude of the vorticity to increase. In the presence of viscosity and no external forces, the reverse happens and energy dissipates. In this scenario, which is called relaxation of the fluid, vortex tubes grow continuously "fatter" as vorticity decreases.

Definition 2.3. *Given a bounded region K of volume V with vorticity field $\boldsymbol{\omega}(\mathbf{x},t)$, the kinetic energy $E_K(K)$ and the energy contained in the vorticity field $E_V(K)$ are defined by*

$$E_K(K) = \frac{1}{2}\int_K \rho \mathbf{v}^2 \, dV$$

$$E_V(K) = \frac{1}{2}\int_K \boldsymbol{\omega}^2 \, dV.$$

The total energy is then

$$E(K) = E_K(K) + E_V(K).$$

Say $\mathbf{j} = \mathbf{v} \times \boldsymbol{\omega}$. Then $\mathbf{j}$, $\mathbf{v}$, and $\boldsymbol{\omega}$ are all mutually perpendicular, and so $|\mathbf{j}| = |\mathbf{v}||\boldsymbol{\omega}|$. It follows that $\boldsymbol{\omega} \times \mathbf{j} = |\boldsymbol{\omega}|^2\mathbf{v}$, since the cross product is a vector in the direction of $\mathbf{v}$ and of magnitude $|\boldsymbol{\omega}||\mathbf{j}|$. Therefore $\nabla \cdot (\boldsymbol{\omega} \times (\mathbf{v} \times \boldsymbol{\omega})) = |\boldsymbol{\omega}|^2(\nabla \cdot \mathbf{v}) = 0$. Now, using this fact, properties of vector calculus, and the frozen field equation for vorticity (3), we obtain:

$$\begin{aligned}
\frac{dE_V}{dt}(K) &= \frac{1}{2}\int_K 2\boldsymbol{\omega} \cdot \frac{\partial \boldsymbol{\omega}}{\partial t} \, dV \\
&= \int_K \boldsymbol{\omega} \cdot (\nabla \times (\mathbf{v} \times \boldsymbol{\omega})) \, dV \\
&= \int_K (\nabla \times \boldsymbol{\omega}) \cdot (\mathbf{v} \times \boldsymbol{\omega}) + \nabla \cdot (\boldsymbol{\omega} \times (\mathbf{v} \times \boldsymbol{\omega})) \, dV \\
&= \int_K \mathbf{v} \cdot (\boldsymbol{\omega} \times (\nabla \times \boldsymbol{\omega})) \, dV \\
&= -\int_K \mathbf{v} \cdot \mathbf{F} \, dV.
\end{aligned}$$

This means that the rate of change of $E_V(K)$ is the negation of the work done by the force $\mathbf{F}$. This force will always have the same direction as velocity, because $\mathbf{v}$ determines the direction of the vorticity field $\boldsymbol{\omega}$ which generates $\mathbf{F}$. Therefore work is positive, and hence $\frac{dE_V}{dt}(K)$ is necessarily always negative. We conclude that the energy of the vorticity field is monotonically decreasing.

Note that because $\nabla \cdot \mathbf{v} = 0$,

$$\begin{aligned}
\mathbf{v} \cdot \nabla^2 \mathbf{v} &= \mathbf{v} \cdot [\nabla(\nabla \cdot \mathbf{v}) - \nabla \times (\nabla \times \mathbf{v})] \\
&= \mathbf{v} \cdot (-\nabla \times \boldsymbol{\omega}) \\
&= \nabla \cdot (\mathbf{v} \times \boldsymbol{\omega}) - (\nabla \times \mathbf{v}) \cdot \boldsymbol{\omega} \\
&= \nabla \cdot (\mathbf{v} \times \boldsymbol{\omega}) - \boldsymbol{\omega}^2.
\end{aligned}$$

Applying the Navier-Stokes equation for conservation of momentum (2) and the Divergence

Theorem, we now have:

$$
\begin{aligned}
\frac{dE_K}{dt}(K) = & \ \frac{1}{2}\int_K 2\mathbf{v}\cdot\rho\frac{\partial\mathbf{v}}{\partial t}\,dV \\
= & \int_K \mathbf{v}\cdot\left(-\rho(\mathbf{v}\cdot\nabla\mathbf{v})+\nabla p+\mathbf{F}+\mu\nabla^2\mathbf{v}\right)\,dV \\
= -\rho & \int_K \mathbf{v}^2(\nabla\mathbf{v})\,dV+\int_K \nabla p\cdot\mathbf{v}\,dV+\int_K \mathbf{v}\cdot\mathbf{F}\,dV+\mu\int_K \nabla\cdot(\mathbf{v}\times\boldsymbol{\omega})\,dV-\mu\int_K \boldsymbol{\omega}^2\,dV \\
= -\rho & \int_{\partial K}\mathbf{v}^2(\mathbf{v}\cdot\mathbf{n})\,dS+\int_{\partial K} p(\mathbf{v}\cdot\mathbf{n})\,dS+\mu\int_{\partial K}(\mathbf{v}\times\boldsymbol{\omega})\cdot\mathbf{n}\,dS+\int_K \mathbf{v}\cdot\mathbf{F}\,dV-\mu\int_K \boldsymbol{\omega}^2\,dV,
\end{aligned}
$$

where $\mathbf{n}$ is the unit normal to the boundary surface of K. If K is a closed, possibly knotted vortex tube, then $\boldsymbol{\omega}=0$ and $\mathbf{v}\cdot\mathbf{n}=0$ on its boundary ∂K, meaning that each of the above surface integrals vanishes. The result is the following set of equations:

$$\frac{dE_K}{dt}(K)=\int_K \mathbf{v}\cdot\mathbf{F}\,dV-\mu\int_K \boldsymbol{\omega}^2\,dV. \tag{4}$$

$$\frac{dE_V}{dt}(K)=-\int_K \mathbf{v}\cdot\mathbf{F}\,dV. \tag{5}$$

$$\frac{dE}{dt}(K)=-\mu\int_K \boldsymbol{\omega}^2\,dV. \tag{6}$$

Therefore, for a nonzero velocity, there are two components to the transfer of energy: the energy of the vortex field is converted to kinetic energy, and kinetic energy is lost to heat. Both $E_V(K)$ and $E(K)$ are monotonically decreasing. In the absence of topological barriers, all energy is eventually transferred to heat, leaving the fluid at rest. However, if K is knotted, the structure of the knot limits the dissipation of energy. We will prove the existence of a positive lower bound for the energy of a nontrivial K, but first must review the necessary knot theory.

3 Helicity as a Lower Bound

3.1 The Călugăreanu-White Linking Number

Any closed curve that is not self-intersecting is a knot and hence admits an infinite number of knot diagrams. A knot diagram is any two-dimensional projection which distinguishes between over-crossings and under-crossings of the knot (see Figure 3), and two knots are equivalent if one can be deformed to the other continuously and without self-intersection. A knot invariant is any property that holds under all such deformations, and thus is the same for every possible diagram of a given knot. The same definitions apply to collections of non-intersecting knots, called links. One such link invariant is the Gauss linking number.

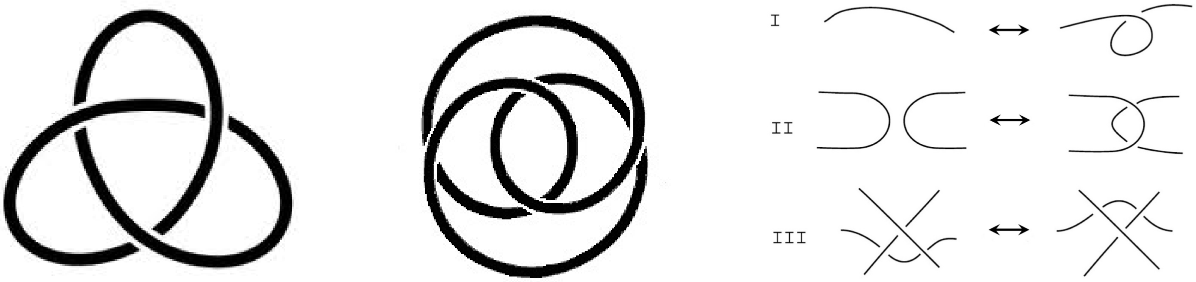

Figure 3: Equivalent knot diagrams representing the trefoil knot (left, center). Two diagrams are equivalent if and only if one can be deformed to the other by some combination of the three Reidemeister moves (right). Source: spie.org.

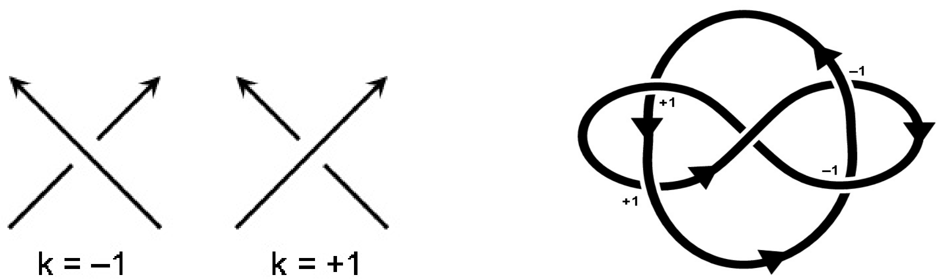

Figure 4: Crossings are assigned a value based on relative orientation (left). The linking number of the Whitehead link (right) is 0. Source: inspirehep.net.

Definition 3.1. *Given any diagram of two linked, oriented curves K_1 and K_2, the Gauss linking number of K_1 and K_2 is given by*

$$Lk(K_1, K_2) = \frac{1}{2}\sum_{i=1}^{n} k_i,$$

where $k_i = \pm 1$ is a value determined by the relative orientations of the top and bottom strands at the ith crossing of K_1 and K_2 (see Figure 4). Equivalently (Nipoti & Ricca [16]), if we parameterize the curves by $K_1 = x(s)$ and $K_2 = y(t)$, then

$$Lk(K_1, K_2) = \frac{1}{4\pi}\int_{K_1 \times K_2} \left(\frac{dx(s)}{ds} \times \frac{dy(t)}{dt}\right) \cdot \frac{(x(s) - y(t))}{|x(s) - y(t)|^3}\, ds\, dt.$$

This quantity is invariant because if we alter a given projection of a link so as to add or remove a crossing between the two knots, we must add or remove both a positive and a negative crossing, meaning that the net change to the Gauss linking number is 0. However, if we try to define a similar invariant for a single knot K^* by looking at its self-crossings, a problem arises: we can add or remove a single crossing to the knot diagram by putting in or taking out a loop (Reidemeister move I, Figure 3). This alters the overall sum of the crossing values, which we call the writhe of the diagram. Since this definition of writhe can be inconsistent for equivalent diagrams, we will instead use the Călugăreanu-White linking number. In determining this number we must first define the writhing number and twist of a knot.

Definition 3.2. *The writhing number $Wr(K^*)$ of a knot K^* is the average writhe of every knot diagram admitted by K^*. Equivalently, using the parameterization $K^* = x(s)$,*

$$Wr(K^*) = Lk(K^*, K^*) = \frac{1}{4\pi} \int_{K^* \times K^*} \left(\frac{dx(s)}{ds} \times \frac{dx(t)}{dt} \right) \cdot \frac{(x(s) - x(t))}{|x(s) - x(t)|^3} \, ds \, dt.$$

The writhing number of a knot measures the extent to which it coils around itself.

Now for $K^* = x(s)$, we parameterize a second curve $y(s) = x(s) + \epsilon u(s)$, where ϵ is constant and $u(s)$ is a unit vector perpendicular to the tangent of x at s. Together, these two curves form the boundary of a ribbon. In addition to the coiling of K^*, we can consider the twist of the ribbon, which is defined as follows:

Definition 3.3. *The twist $Tw(K^*)$ of a ribbon bounded by $x(s)$ and $y(s)$ is the average winding of the curve $y(s)$ around $x(s)$, given by*

$$Tw(K^*) = \frac{1}{2\pi} \int_{K^*} \left(u(s) \times \frac{du(s)}{ds} \right) \cdot \frac{dx(s)}{ds} \, ds.$$

Though the writhing number and twist are dependent upon the framing of K^*, their sum is invariant.

Definition 3.4. *The Călugăreanu-White linking number $Lk(K^*)$ of a knot K^* is the sum of the writhing number and twist of K^*,*

$$Lk(K^*) = Wr(K^*) + Tw(K^*).$$

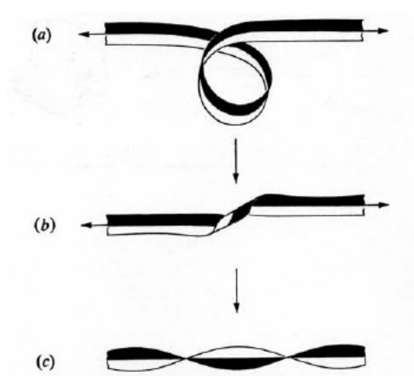

Figure 5: Deformation of (a) a component of the writhe of a ribbon so that it becomes (c) a component of twist. Though the writhe and twist are being altered, their sum is constant. The intermediate stage (b) depicts torsion, which contributes to twist and will be discussed in section 5. Source: Moffatt & Ricca [14].

It is important to note that the writhing number and twist of a knot are invariant under most types of deformation, including stretching and shortening of the knot. It is only when a loop is added or removed, which involves the formation of an inflection point, that these values are altered. In Section 5 we will ignore the possibility of inflection points and use twist as an invariant.

3.2 Helicity of a Vortex Tube

The axis K^* of a closed, possibly knotted vortex tube K is a knot that is equivalent in type to the tube itself. The tube, and hence its axis, is oriented based on the direction of the vorticity, and any diagram of K^* will inherit this orientation. The writhing number and twist of K are then $Wr(K) = Wr(K^*)$ and $Tw(K) = Tw(K^*)$, where the normal u used to determine twist connects the axis to some closed vortex line in ∂K. Because $Lk(K) = Wr(K)+Tw(K)$ is a knot invariant and vorticity is a frozen field under the Navier-Stokes equations, $Lk(K)$ is invariant under the Navier-Stokes equations. The Gauss linking number of a pair of linked vortex tubes is the Gauss linking number of their respective axes, also an invariant under Navier-Stokes.

Definition 3.5. *Let L be a link comprised of vortex tubes $K_1, K_2, ..., K_n$ with circulations $\Phi_1, \Phi_2, ..., \Phi_n$, respectively, with velocity field $\mathbf{v}$ and vorticity $\boldsymbol{\omega}$. Then the helicity of L is*

$$H(L) = \int_L \mathbf{v} \cdot \boldsymbol{\omega} \quad dV$$

or, equivalently,

$$H(L) \;=\; \sum_{i=1}^{n} \Phi_i^2 \, Lk(K_i) \;+\; 2 \sum_{1 \leq i < j \leq n} \Phi_i \Phi_j \, Lk(K_i, K_j).$$

Helicity is a well-known invariant of incompressible, homogenous fluid motion, and in non-trivial cases may be used as a lower bound for the energy of a vorticity field.

Theorem 3.6. *For a collection of linked vortex tubes L, there exists a positive constant q such that $E_V(L) \geq q|H(L)|$.*

Proof. By the Schwartz Inequality,

$$\int_L \mathbf{v}^2 dV \int_L \boldsymbol{\omega}^2 dV \;\;\geq\;\; \left| \int_L \mathbf{v} \cdot \boldsymbol{\omega} \, dV \right|^2 \;\;=\;\; |H(L)|^2,$$

and by an adaptation of the Poincare Inequality (Moffatt & Ricca [10]) there exists a constant q_0, which is uniquely determined for the given domain, such that

$$\frac{\int_L \boldsymbol{\omega}^2 dV}{\int_L \mathbf{v}^2 dV} \geq q_0^2.$$

If we multiply these two equations, we obtain

$$\left(\int_L \boldsymbol{\omega}^2 dV \right)^2 \geq q_0^2 |H(L)|^2,$$

which, letting $q = \frac{|q_0|}{2}$, then simplifies to

$$E_V(L) = \frac{1}{2} \int_L \boldsymbol{\omega}^2 dV \geq q|H(L)|. \tag{7}$$

$\square$

Thus the energy of a collection of knotted vortex tubes with nonzero helicity is bounded away from zero. However, there are many nontrivial knots and links with zero helicity, such as the Whitehead link depicted in Figure 4, so this estimate is not sufficient. We will now restrict ourselves to looking at a single knotted vortex tube K, which has helicity $H(K) = \Phi_i^2\, Lk(K_i)$, and will find a lower bound that is nonzero for every nontrivial knot.

For a more thorough discussion of helicity and the Călugăreanu-White linking number, see Dennis & Hannay [3] and Moffatt & Ricca [14].

4 Asymptotic Crossing Number as a Lower Bound

The goal of this section is to prove the existence of a positive lower energy bound which depends purely on the knot type of a vortex tube. The section's primary theorem is as follows:

Theorem 4.1. *Given a knotted vortex tube K of volume V,*

$$E_V(K) \geq \left(\frac{2}{\pi}\right)^{1/3} V^{-1/3}\, c_K(\boldsymbol{\omega}, \boldsymbol{\omega}).$$

Here $c_K(\boldsymbol{\omega}, \boldsymbol{\omega})$ is the asymptotic crossing number of the vorticity field in the domain K, which we will define below.

4.1 Background

The Gauss and Călugăreanu-White linking numbers are insufficient as lower bounds because positively and negatively signed crossings may cancel, resulting in a value of zero for a nontrivial knot. Instead, we will use an adaptation called the crossing number, which adds the absolute value $|k_i| = 1$ at each crossing. We define the average crossing number $c(K_1, K_2)$ of two knots K_1 and K_2 as the average number of over-crossings of K_1 and K_2 among all possible planar projections, which can be calculated using the formula

$$c(K_1, K_2) = \frac{1}{4\pi} \int_{K_1 \times K_2} \left| \left(\frac{dx(s)}{ds} \times \frac{dy(t)}{dt} \right) \cdot \frac{(x(s) - y(t))}{|x(s) - y(t)|^3} \right| \, ds\, dt,$$

where $x(s)$ and $y(t)$ parameterize K_1 and K_2, respectively. This definition also applies to the average crossing number of a single knot, $c(K^*) = c(K^*, K^*)$. Since crossing number disregards orientation and assigns each crossing a positive value, it must be positive for any nontrivial knot and $c(K_1, K_2) \geq Lk(K_1, K_2)$ for all K_1 and K_2. We can extend the definition to vector fields by considering the crossing numbers averaged over all field lines, as follows:

Definition 4.2. *The asymptotic crossing number of any two vector fields W_1 and W_2 over a domain D is given by*

$$c_D(W_1, W_2) = \frac{1}{4\pi} \int_{D \times D} \frac{|(W_1(x) \times W_2(y)) \cdot (x - y)|}{|x - y|^3} \, dV_x dV_y.$$

The asymptotic crossing number, like the Gauss and Călugăreanu-White linking numbers, is a topological invariant that becomes an invariant of the Navier-Stokes equations when applied to a fluid field. For a complete derivation of the asymptotic crossing number and proof of its invariance, see Freedman & He [5].

We now compute the asymptotic crossing number of our knotted vortex tube K:

$$c_K(\boldsymbol{\omega},\boldsymbol{\omega}) = \frac{1}{4\pi}\int_{K\times K} \frac{|(\boldsymbol{\omega}(x)\times\boldsymbol{\omega}(y))\cdot(x-y)|}{|x-y|^3}\,dV_x\,dV_y. \tag{8}$$

Furthermore, note that

$$c_K(\boldsymbol{\omega},\boldsymbol{\omega}) \leq \frac{1}{4\pi}\int_{K\times K} |\boldsymbol{\omega}(x)|\frac{|\boldsymbol{\omega}(y)|}{|x-y|^2}\,dV_y\,dV_x. \tag{9}$$

In the following subsection we will make use of several known inequalities. They are stated here without proof. We use the notation $||f||_p = \left(\int_D |f|^p\,dV\right)^{1/p}$, where $||f|| = ||f||_1$ and D is the particular domain being considered; in our proof $D = K$.

Definition 4.3. *The Riesz potential $I_\alpha f$ of a locally integrable function f on $\mathbb{R}^n$, with constant c determined by the values of n and α, is*

$$I_\alpha f(x) = \frac{1}{c}\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy.$$

Theorem 4.4. *(Hardy-Littlewood-Sobolev Inequality)*[1] *For $0<\alpha<n$ and $1<p<\frac{n}{\alpha}$, let $p^* = \frac{np}{n-\alpha p}$. Then for a locally integrable function f on $\mathbb{R}^n$,*

$$||cI_\alpha f||_{p^*} \leq C||f||_p$$

where

$$C = \pi^{(n-\alpha)/2}\,\frac{\Gamma(\alpha/2)}{\Gamma((n+\alpha)/2)}\left(\frac{\Gamma(n)}{\Gamma(n/2)}\right)^{\alpha/n}.$$

Note that the Gamma function satisfies $\Gamma(n) = (n-1)!$ and $\Gamma(\frac{1}{2}+n) = \frac{(2n)!}{4^n n!}\sqrt{\pi}$ for $n\in\mathbb{N}$.

Theorem 4.5. *(Hölder's Inequality) If f and g are real- or complex-valued functions and $p,q\in\mathbb{R}$ such that $\frac{1}{p}+\frac{1}{q}=1$, then $||fg|| \leq ||f||_p\,||g||_q$.*

We are now ready to complete our proof.

4.2 Proof of Theorem 4.1

Proof. First, using Hölder's Inequality, we obtain

$$\begin{aligned}
\left(||\boldsymbol{\omega}||_{3/2}\right)^2 &= ||\boldsymbol{\omega}^{3/2}||^{4/3}\\
&\leq \left(||1||_4\,||\boldsymbol{\omega}^{3/2}||_{4/3}\right)^{4/3}\\
&= \left(\left(\int_K dV\right)^{1/4}\left(\int_K |\boldsymbol{\omega}|^2\,dV\right)^{3/4}\right)^{4/3}\\
&= V^{1/3}\cdot 2E_V(K),
\end{aligned}$$

[1]For a proof of the Hardy-Littlewood-Sobolev Inequality, see Frank & Lieb, [4].

and therefore

$$E_V(K) \geq \frac{1}{2}\, V^{-1/3} \left(||\boldsymbol{\omega}||_{3/2}\right)^2. \tag{10}$$

Because we are working in $\mathbb{R}^3$, the Riesz potential of $|\boldsymbol{\omega}|$ with $\alpha = 1$ is

$$I_1\boldsymbol{\omega}(x) = \frac{1}{c} \int_K \frac{|\boldsymbol{\omega}(y)|}{|x-y|^2}\, dV_y. \tag{11}$$

If we let $p = \frac{3}{2}$, then $p^* = \frac{np}{n-\alpha p} = 3$, and the Hardy-Littlewood-Sobolev Inequality gives $C||\boldsymbol{\omega}||_{3/2} \geq ||cI_1\boldsymbol{\omega}||_3$. It follows from this and Hőlder's Inequality that

$$\begin{aligned} C||\boldsymbol{\omega}||_{3/2}^2 &\geq ||\boldsymbol{\omega}||_{3/2}\, ||cI_1\boldsymbol{\omega}||_3 \\ &\geq \int_K |\boldsymbol{\omega}(x)|\, |cI_1\boldsymbol{\omega}(x)|\, dV_x \\ &= \int_{K\times K} |\boldsymbol{\omega}(x)| \int_K \frac{|\boldsymbol{\omega}(y)|}{|x-y|^2}\, dV_y\, dV_x \\ &\geq 4\pi\, c_K(\boldsymbol{\omega}, \boldsymbol{\omega}). \end{aligned}$$

We find the constant C as follows:

$$\begin{aligned} C &= \pi \frac{\Gamma(1/2)}{\Gamma(2)} \left(\frac{\Gamma(3)}{\Gamma(3/2)}\right)^{1/3} \\ &= \pi \left(\frac{\pi^{1/2}}{1}\right) \left(\frac{2}{(1/2)\pi^{1/2}}\right)^{1/3} \\ &= 2^{2/3}\, \pi^{4/3} \end{aligned}$$

Continuing from equation (10),

$$\begin{aligned} E_V(K) &\geq \frac{1}{2}\, V^{-1/3} \left(\frac{4\pi}{C}\, c_K(\boldsymbol{\omega}, \boldsymbol{\omega})\right) \\ &= \left(\frac{2}{\pi}\right)^{1/3} V^{-1/3}\, c_K(\boldsymbol{\omega}, \boldsymbol{\omega}). \end{aligned} \tag{12}$$

□

Therefore, for any nontrivial K, $E_V(K)$ is bounded away from zero. Furthermore, Freedman & He [5] proved that $c_K(\boldsymbol{\omega}, \boldsymbol{\omega}) \geq 2\,\text{genus}(K) - 1$, where $\text{genus}(K)$ is the minimal genus of any compact, connected, oriented surface whose boundary is a knot of type K^*. The genus of such a surface, called a Seifert surface, represents the maximum number of non-intersecting simple closed curves that can be drawn on the surface without separating it into disconnected components. The genus of a knot is a knot invariant which is zero only for the trivial case, where K^* is the unknot. Otherwise it is a positive integer, and hence $2\,\text{genus}(K) - 1 > 0$.

5 Estimating the Energy Minimum

Because the energy of a vortex tube is monotonically decreasing and bounded away from zero, it must converge to a positive limit. In this section we will use a non-orthogonal coordinate system to determine the energy purely as a function of topological invariants and the length of the vortex tube. We then conclude that convergence to the energy minimum is accompanied by either stretching or shortening of the vortex tube, as determined by volume and knot type. Finally, we narrow our focus to vortex tubes that form torus knots.

In order to achieve the desired results we will restrict the discussion to vortex tubes that are uniform along their length and of circular cross-section. The expression so obtained provides an upper bound for the energy of any vortex tube that does not satisfy these conditions (Chui & Moffatt, [2]). We will also presume a standard toroidal and poloidal flux (defined below) and the absence of singularities, discontinuities, or inflection points during the deformation process.[2]

5.1 Definition of a Non-Orthogonal Coordinate System

We begin by defining an appropriate coordinate system. We parameterize the axis K^* of the knotted vortex tube K by $\mathbf{X}(s)$, where s represents arc length from an arbitrary starting point on the curve, L is the length of K^*, and V is the volume of K. Let

$c(s)$ = the curvature
$\tau(s)$ = the torsion
$\mathbf{t}(s)$ = the unit tangent
$\mathbf{n}(s)$ = the unit principal normal
$\mathbf{b}(s)$ = the unit binormal; $\mathbf{b} = \mathbf{t} \times \mathbf{n}$

The curvature of K^* is the magnitude of the rate of change of the unit tangent, and the torsion is the magnitude of the rate of change of the unit binormal. They are defined by the Frenet equations, as follows:

$$\frac{d\mathbf{t}}{ds} = c\mathbf{n}, \qquad \frac{d\mathbf{n}}{ds} = -c\mathbf{t} + \tau\mathbf{b}, \qquad \frac{d\mathbf{b}}{ds} = -\tau\mathbf{n}. \tag{13}$$

The vortex lines in K form a family of distinct nested surfaces S_χ, each bounding a tubular neighborhood of K^*, where χ ranges from 0 to 1 so that $S_0 = K^*$ and $S_1 = \partial K$. Then we let

V_χ = the volume of the area bounded by S_χ
$\chi = V/V_\chi$
θ = an angle measured from $\mathbf{n}$ in the plane spanned by $\mathbf{n}$ and $\mathbf{b}$
Γ_θ = the intersection of ∂K and $\mathbf{X}(s) + r\cos\theta\mathbf{n}(s) + r\sin\theta\mathbf{t}(s)$ for a fixed θ
N_θ = the Gauss linking number of K^* with the curve Γ_θ
$\phi = \theta + 2\pi N_\theta s/L$
$\tau^* = \tau - 2\pi N_\theta/L$
$r = R(s, \chi, \phi)$ = the distance between the point $\mathbf{X}(s)$ and S_χ in the direction of ϕ

[2]The coordinate system used in this section is not suited to working with inflection points. The possibility of inflection points is discussed in Moffatt & Ricca [15].

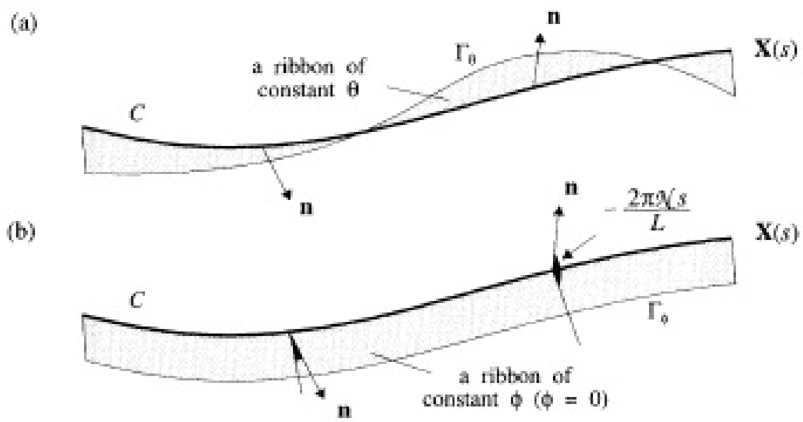

Figure 6: For a fixed θ, (a) the ribbon bounded by K^* and Γ_θ, and (b) the ribbon bounded by K^* and Γ_ϕ. Source: Chui & Moffatt [2].

Any point in K may be represented by traveling some distance along K^* and then, with the appropriate angle and radius, moving within a cross-section of K. Thus we adopt the coordinate system (s, χ, ϕ) and, noting that s and ϕ are periodic with periods L and 2π, respectively, we can represent any point $\mathbf{x}$ in K^* by

$$\begin{aligned} \mathbf{x} &= \mathbf{X}(s) + r\cos\theta\mathbf{n}(s) + r\sin\theta\mathbf{b}(s) \\ &= \mathbf{X}(s) + R(s,\chi,\phi)\cos(\phi - 2\pi N_\theta s/L)\mathbf{n}(s) + R(s,\chi,\phi)\sin(\phi - 2\pi N_\theta s/L)\mathbf{b}(s). \end{aligned} \tag{14}$$

We choose ϕ in place of θ in the interest of zero-framing. For a fixed ϕ and $\chi = 1$, the ribbon bounded by K^* and the curve $\mathbf{x}(s) = (\mathbf{X}(s) + r\cos\theta\,\mathbf{n} + r\sin\theta\,\mathbf{b})$ has zero twist (see Figure 6). Furthermore, because we are assuming that the vortex tube is uniform along its length and of circular cross-section, S_χ has a constant radius of $R(\chi)$ for any χ. In other words, $R(s, \chi, \phi)$ is independent of s and ϕ. Then $V_\chi = \pi R(\chi)^2$Ł, and hence, letting $\epsilon = \left(\frac{V}{\pi L^3}\right)^{1/2}$,

$$R(\chi) = \epsilon L \chi^{1/2}. \tag{15}$$

We now write

$$d\mathbf{x} = \mathbf{e_1}ds + \mathbf{e_2}d\chi + \mathbf{e_3}d\phi$$

with basis vectors and the resulting metric tensor as follows:

$$\begin{aligned} \mathbf{e_1} &= (1 - Rc\cos\theta)\mathbf{t} - R\tau^*\sin\theta\mathbf{n} + R\tau^*\cos\theta\mathbf{b} \\ \mathbf{e_2} &= R'\,(\cos\theta\,\mathbf{n} + \sin\theta\,\mathbf{b}) \\ \mathbf{e_3} &= -R\sin\theta\,\mathbf{n} + R\cos\theta\,\mathbf{b} \end{aligned}$$

$$\begin{aligned} (g_{ij}) = (e_i, e_j) &= \begin{pmatrix} (1 - Rc\cos\theta)^2 + R^2\tau^{*2} & 0 & R^2\tau^* \\ 0 & (R')^2 & 0 \\ R^2\tau^* & 0 & R^2 \end{pmatrix} \\ &= \begin{pmatrix} (1 - \epsilon L\chi^{1/2}c\cos\theta)^2 + \epsilon^2L^2\chi\tau^{*2} & 0 & \epsilon^2L^2\chi\tau^* \\ 0 & \epsilon^2L^2/4\chi & 0 \\ \epsilon^2L^2\chi\tau^* & 0 & \epsilon^2L^2\chi \end{pmatrix} \end{aligned} \tag{16}$$

where $R' = \frac{dR}{d\chi}$. We denote by g the determinant of the matrix, and finally obtain the Jacobian of our transformation:

$$\begin{aligned} J = \sqrt{g} &= RR'(1 - Rc\cos\theta) \\ &= \frac{1}{2}\epsilon^2 L^2 (1 - \epsilon L \chi^{1/2} c \cos\theta). \end{aligned} \tag{17}$$

This is well-defined with the added assumption that K has sufficiently small cross-sections to satisfy $Rc < 1$ at all points.

5.2 Energy Minimization

We now return to the vorticity field $\boldsymbol{\omega}$ in the context of our new coordinate system. Because the surfaces S_χ are comprised of vortex lines, $\boldsymbol{\omega} \cdot \nabla\chi = 0$ and hence

$$\boldsymbol{\omega} = \omega_1 \mathbf{e_1} + \omega_3 \mathbf{e_3}. \tag{18}$$

Furthermore, $\nabla \cdot \boldsymbol{\omega} = 0$ implies

$$\frac{\partial}{\partial s}(\sqrt{g}\, \omega_1) + \frac{\partial}{\partial \phi}(\sqrt{g}\, \omega_2) = 0. \tag{19}$$

It follows that there exists a flux function $\psi(s, \chi, \phi)$ satisfying

$$\omega_1 = -\frac{1}{\sqrt{g}} \frac{\partial \psi}{\partial \phi}, \qquad \omega_2 = \frac{1}{\sqrt{g}} \frac{\partial \psi}{\partial s}. \tag{20}$$

Let $T(\chi)$ denote the flux across any cross section of S_χ, called the toroidal flux, and $P(\chi)$ the poloidal flux across any ribbon determined by a constant ϕ and bounded by K^* and S_χ. Then, by Chui & Moffatt [2], we can decompose ψ into

$$\psi(s, \chi, \phi) = \tilde{\psi}(s, \chi, \phi) - \frac{\phi}{2\pi} \frac{dT}{d\chi} + \frac{s}{L} \frac{dP}{d\chi} \tag{21}$$

where $\tilde{\psi}$ is a single-valued, periodic function with a period of L in s and a period of 2π in ϕ. Due to the assumption of uniformity throughout the vortex tube, ψ is independent of s. We will now adopt the convention for the toroidal and poloidal fluxes of a standard flux tube: $P' = hT' = h\Phi$, where $h = Lk(K) = \frac{H(K)}{\Phi^2}$ (Chui & Moffatt [2]). Thus, letting

$$W_1 = \sqrt{g}\, \omega_1 = -\frac{\partial \tilde{\psi}}{\partial \phi} + \frac{\Phi}{2\pi}, \qquad W_3 = \sqrt{g}\, \omega_3 = \frac{h\Phi}{L}, \tag{22}$$

the energy of the vortex tube is

$$\begin{aligned} E_V(K) &= \frac{1}{2} \int_K \boldsymbol{\omega}^2 \, dV \\ &= \frac{1}{2} \iiint_K \sqrt{g}\, (\omega_1 \mathbf{e_1} + \omega_3 \mathbf{e_3})^2 \, ds\, d\chi\, d\phi \\ &= \frac{1}{2} \iiint_K \frac{1}{\sqrt{g}} \left(W_1^2\, g_{11} + 2W_1 W_3\, g_{13} + W_3^2\, g_{33}\right) ds\, d\chi\, d\phi. \end{aligned} \tag{23}$$

Now we utilize the fact that $\epsilon = \left(\frac{V}{\pi L^3}\right)^{1/2} \ll 1$. We will determine the energy as a power series expansion in terms of ϵ and, due to the smallness of ϵ, may disregard any terms with a positive degree. Note that neither W_1 nor W_2 is dependent upon s, and also that

$$\begin{aligned}
\tfrac{g_{11}}{\sqrt{g}} &= \tfrac{2}{\epsilon^2 L^2}(1 - \epsilon\chi^{1/2} c\cos\theta + \epsilon^2 L^2 \chi \tau^{*2} + \ldots) \\
\tfrac{g_{13}}{\sqrt{g}} &= 2\chi\tau^*(1 + \epsilon\chi^{1/2} c\cos\theta + \ldots) \\
\tfrac{g_{13}}{\sqrt{g}} &= 2\chi(1 + \epsilon\chi^{1/2} c\cos\theta + \ldots).
\end{aligned}$$

We will use the trigonometric identity $\cos\theta = \cos\phi\cos(2\pi N_\theta s/L) - \sin\phi\sin(2\pi N_\theta s/L)$ and let

$$a_N = \int_0^L c(s)\sin(2\pi N_\theta s/L)\,ds, \qquad b_N = \int_0^L c(s)\cos(2\pi N_\theta s/L)\,ds.$$

Therefore,

$$\begin{aligned}
&\iiint_K W_1^2 \frac{g_{11}}{\sqrt{g}}\,ds\,d\chi\,d\phi \\
&= \int_0^1\int_0^{2\pi} W_1^2\left(\int_0^L \frac{g_{11}}{\sqrt{g}}\,ds\right) d\phi\,d\chi \\
&\approx \int_0^1\int_0^{2\pi} \frac{2}{\epsilon^2 L^2} W_1^2\left(L + \epsilon\chi^{1/2}(a_N\sin\phi - b_N\cos\phi) + \epsilon^2 L^2\chi\oint\tau^{*2}ds\right) d\phi\,d\chi. \qquad (24)
\end{aligned}$$

Because $\tilde{\psi}(\chi,\phi)$, $\cos\phi$, and $\sin\phi$ all have a period of 2π in ϕ, and all other terms are independent of ϕ, (24) evaluates to:

$$\begin{aligned}
\iiint_K \frac{g_{11}}{\sqrt{g}} W_1^2\,ds\,d\chi\,d\phi &\approx \int_0^1 \frac{2}{\epsilon^2 L^2}\left(\frac{\Phi}{2\pi}\right)^2\left(L + \epsilon^2 L^2\chi\oint\tau^{*2}ds\right)(2\pi)\,d\chi. \\
&= \frac{\Phi^2}{\pi\epsilon^2 L^2}\int_0^1\left(L + \epsilon^2 L^2\chi\oint\tau^{*2}ds\right)d\chi \\
&= \frac{\Phi^2}{\pi\epsilon^2 L^2}\left(L + \frac{1}{2}\epsilon^2 L^2\oint\tau^{*2}ds\right). \qquad (25)
\end{aligned}$$

The same methods yield

$$\begin{aligned}
&\iiint_K \left(2\,W_1 W_3\frac{g_{13}}{\sqrt{g}} + W_3^2\frac{g_{33}}{\sqrt{g}}\right) ds\,d\chi\,d\phi \\
&= \int_0^1\int_0^{2\pi} 2\,W_1 W_3\left(\int_0^L \frac{g_{13}}{\sqrt{g}}\,ds\right) + W_3^2\left(\int_0^L \frac{g_{33}}{\sqrt{g}}\,ds\right) d\phi\,d\chi \\
&\approx \int_0^1 \left(2\left(\frac{h\Phi^2}{2\pi L}\right)(2\chi L\oint\tau^* ds) + \left(\frac{h^2\Phi^2}{L^2}\right)(2\chi L)\right)(2\pi)\,d\chi \\
&= \Phi^2\left(\frac{h}{\pi}\oint\tau^* ds + \frac{h^2}{L}\right). \qquad (26)
\end{aligned}$$

Thus, substituting these results into (23) and neglecting all positive powers of ϵ, we obtain

$$\begin{aligned} E_V(K) &\approx \frac{\Phi^2}{2\pi L}\,\epsilon^{-2} + \frac{\Phi^2}{4\pi L}\left(L\oint \tau^{*2}ds + 2hL\oint \tau^* ds + 2\pi h^2\right) \\ &= \frac{\Phi^2}{2V}\,L^2 + \frac{\Phi^2}{4\pi}\left(\oint \tau^{*2}ds + 2h\oint \tau^* ds\right) + \frac{\Phi^2 h^2}{2}L^{-1}. \end{aligned} \tag{27}$$

For a unit normal $\mathbf{u} = \mathbf{n}\cos\theta + \mathbf{b}\sin\theta$, using the Frenet equations and the fact that $\frac{d\theta}{ds} = -\frac{2\pi N_\theta}{L}$, we calculate $\frac{d\mathbf{u}}{ds} = -c\mathbf{t}\cos\theta - \tau^*\mathbf{n}\sin\theta + \tau^*\mathbf{b}\cos\theta$. Then

$$\begin{aligned} Tw(K) &= \frac{1}{2\pi}\oint\left(\mathbf{u}\times\frac{d\mathbf{u}}{ds}\right)\cdot\frac{d\mathbf{x}}{ds}\,ds \\ &= \frac{1}{2\pi}\oint\big(\tau^*\mathbf{t} - (c\sin\theta\cos\theta + \tau^*\cos\theta)\mathbf{n} + c(\cos\theta)^2\mathbf{b}\big)\cdot\mathbf{t}\;ds \\ &= \frac{1}{2\pi}\oint \tau^*\,ds. \end{aligned} \tag{28}$$

This is significant because, ignoring the possibility of inflection points, all quantities in equation (27) except length are invariant under fluid motion. We can therefore obtain the energy minimum by minimizing with respect to length, subject to the constraints of the field topology.

$$\begin{aligned} \frac{dE_V}{dL}(K) &\approx \Phi^2\left(\frac{1}{V}L - \frac{h^2}{2}L^{-2}\right) \\ &= \frac{\Phi^2}{VL^3}\left(L^2 - \frac{Vh^2}{2}\right). \end{aligned}$$

Clearly, the energy is decreasing where $L < \left(\frac{Vh^2}{2}\right)^{1/2}$ and increasing where $L > \left(\frac{Vh^2}{2}\right)^{1/2}$, and so is theoretically minimized where

$$L = \left(\frac{Vh^2}{2}\right)^{1/2}. \tag{29}$$

Recall that because V is constant, a decrease in length corresponds to an increase in cross-sectional area, while tube stretching corresponds to a decrease in cross-sectional area. If h is small, the energy decreases as L decreases, which continues until self-contact prevents further cross-sectional growth. However, for a strongly knotted tube with a large h, the length may initially be below the critical point, in which case tube length increases as the energy is minimized.

For a sufficiently large h, substituting equation (29) into (27) provides a valid minimum:

$$\min(E_V(K)) \approx \Phi^2\left(\frac{h^2}{4} + \frac{1}{4\pi}\oint \tau^{*2}ds + \frac{h}{2\pi}\oint \tau^* ds + \frac{h}{(2V)^{1/2}}\right). \tag{30}$$

However, h may be small enough that the length is prevented by the field topology from decreasing to $\left(\frac{Vh^2}{2}\right)^{1/2}$. Clearly, (30) is invalid for $h = 0$ because the length must be nonzero. These options are depicted in Figure 7.

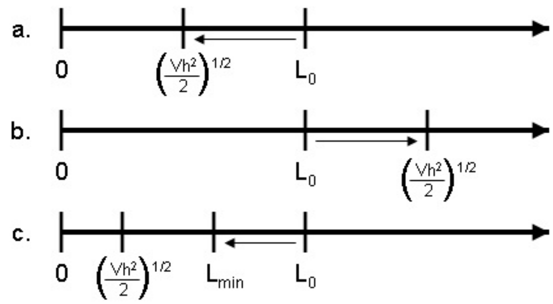

Figure 7: (a) For a sufficiently small h, the initial length L_0 decreases to $\left(\frac{Vh^2}{2}\right)^{1/2}$. (b) For a large h, the initial length L_0 increases to $\left(\frac{Vh^2}{2}\right)^{1/2}$. (c) The initial length L_0 decreases, but a minimum length $L_{min} > \left(\frac{Vh^2}{2}\right)^{1/2}$ is imposed by the field topology.

In the next subsection we will look specifically at the class of knots called torus knots, for which we can identify the minimum length, and hence the minimum energy.

5.3 Torus Knots

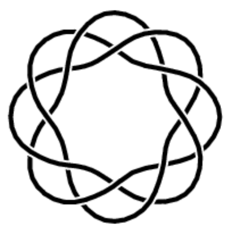

Figure 8: The $T_{3,-8}$ torus knot. Source: wikipedia.org.

Consider a knot K^* that lies on the surface of an unknotted torus T. Then let D be any cross-sectional disk in T and let C be the axis of T. The wrapping number q of K^* is the minimum number of intersections of K^* and D over all allowable deformations of K^* in T. The winding number, or degree, of K^* is $p = Lk(K^*, C)$. If p and q are coprime, we say that K^* is a torus knot of type $T_{p,q}$. The torus knots $T_{p,q}$ and $T_{q,p}$ are equivalent, and negating either p or q yields the mirror image knot. $T_{p,q}$ is trivial if and only if p or q is ± 1.

We can parameterize a torus knot by

$$\mathbf{X}(t) = R((1 + \lambda cos(qt))cos(pt), (1 + \lambda cos(qt))sin(pt), \lambda sin(qt)) \tag{31}$$

where $0 \le t \le 2\pi$. We let $\lambda > 0$ so K^* is left-handed. It follows that the length L of K^* is

$$L = R \int_0^{2\pi} \sqrt{\lambda^2 q^2 + p^2 (1 + \lambda cos(qt))^2}\, dt = R\, \ell(\lambda), \tag{32}$$

where $\ell(\lambda)$ is defined by $\ell(\lambda) = L/R$.

Now let K^*, a torus knot parameterized as above, be the axis of a vortex tube K with uniform circular cross-sections. The radius of the tube is then $R(1) = \epsilon L$ by equation 15. There is a function $\tilde{d}(\lambda)$ that satisfies

$$R\,\tilde{d}(\lambda) = \min |\mathbf{X}(t) - \mathbf{X}(s)|. \tag{33}$$

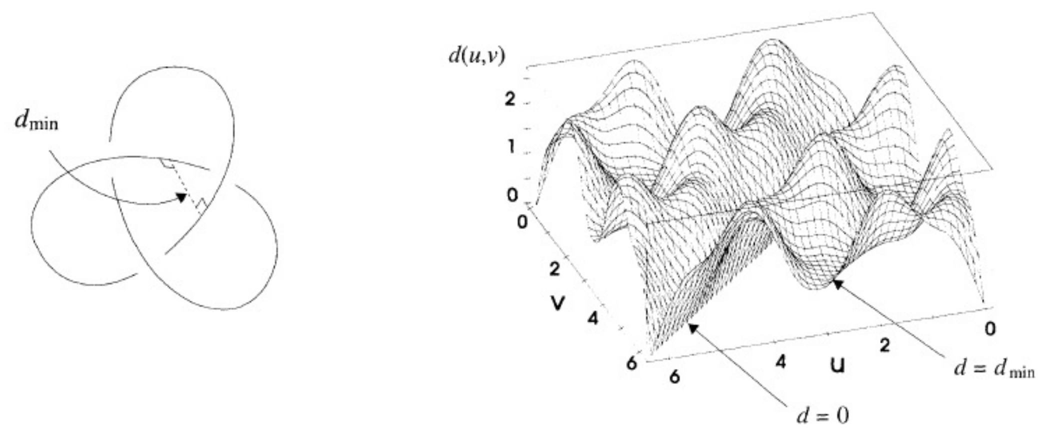

Figure 9: Local minimum d_{min} of the separation function for the trefoil knot $T_{2,3}$ (left). Surface plot of $d(t,s) =$ constant for $T_{2,3}$, where $R = 1$ and $\lambda = .4$ (right). Source: Chui & Moffatt [2].

For a small λ, $\tilde{d}(\lambda) \approx 2\lambda \sin(\pi/p)$. If h is small, the energy of K is minimized when ϵL is greatest, which occurs at half the minimum value of the function $d(t,s) = |\mathbf{X}(t) - \mathbf{X}(s)|$, so we let $\tilde{d}(\lambda) = 2\epsilon L$. This implies

$$R^3 = \frac{4V}{\pi\,\ell(\lambda)(\tilde{d}(\lambda))^2}$$

and

$$L = R\,\ell(\lambda) = \left(\frac{4V}{\pi}\right)^{1/3} \left(\frac{\ell(\lambda)}{\tilde{d}(\lambda)}\right)^{2/3}. \tag{34}$$

Having obtained L as a function of λ, we calculate the energy minimum by minimizing L with respect to λ and then substituting the result into (27). See Chui & Moffatt [2] for further detail and calculations given a variety of specific torus knots.

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