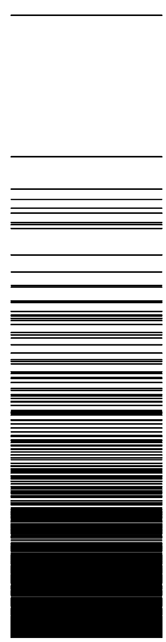
A Glimpse of the Markoff Spectrum

Travis Morrison

March 23, 2012



Abstract

In this paper, we describe an algorithm for producing a plot of part of the Markoff spectrum. We approach the spectrum from the the theory of indefinite binary quadratic forms and develop some of this theory in a visual way, using the range topograph of Conway. This simplifies many of the relevant proofs and gives an elementary algorithm for computing equivalent reduced binary quadratic forms and finding their minima.

I would like to thank Martin Weissman for his extensive guidance throughout the writing of this paper, and my mother and father, for their continued love and support.

Contents

1 Introduction **3**

2 The Topograph **4**
2.1 The Domain Topograph 4
2.2 The Range Topograph 6
2.2.1 Equivalent Binary Quadratic Forms 6
2.2.2 Constructing the Range Topograph 7

3 The Markoff Spectrum **12**

4 Seeing the Spectrum **15**
4.1 The Code 20

1 Introduction

The Markoff spectrum, as a subject of study number theory, is relatively young, having been first investigated by Andrey Markoff in the late 19th century. The subject was further developed in the mid 20th century. In some treatments, the Markoff spectrum is defined via continued fractions, but it has an equivalent definition in terms of binary quadratic forms. We approach the subject with this perspective; we will define the Markov spectrum to be the collection the minimum values (which we normalize in a suitable way) of all binary quadratic forms. In an effort to truly see the nature of the spectrum, we take a visual approach to the binary quadratic form.

A **binary quadratic form** (BQF) is a function $Q : \mathbb{Z}^2 \to \mathbb{Z}$ given by $(x, y) \to ax^2 + bxy + cy^2$ where $a, b, c \in \mathbb{Z}$. In most of the literature on binary quadratic forms, the integers x, y are assumed to be coprime ($\gcd(x, y) = 1$) because we have for any integer $n \in \mathbb{Z}$ and a binary quadratic form $Q(x, y)$, $Q(nx, ny) = n^2 Q(x, y)$, so it suffices to understand what a form does to relatively prime pairs. This is particularly important to our cause, since the calculation shows if a form has a minimum m and $(x, y) \in \mathbb{Z}^2$ is such that $Q(x, y) = m$, then it must be the case that $\gcd(x, y) = 1$.

We will not consider all binary quadratic forms when we construct the Markoff spectrum; only a certain type of form is interesting from this perspective. If $Q(x, y) = ax^2 + bxy + cy^2$ is a binary quadratic form, define the **discriminant** of Q, denoted $\Delta(Q)$ (or just Δ when it is understood what Q is), to be $b^2 - 4ac$. The nature of this number, like its magnitude, sign, or prime factorization, yields a wealth of information about the form. We are now ready to define the type of forms that will be of interest: we say that a binary quadratic form is **indefinite** if $\Delta > 0$. We will further restrict ourselves to the case when $\Delta(Q)$ is not a perfect square, as we will see that this guaruntees Q is nonzero for relatively prime (x, y).

The Markoff spectrum is defined to be the set of all such

$$\frac{1}{\sqrt{\Delta}} \min_{\substack{x,y \in \mathbb{Z} \\ \gcd(x,y)=1}} |Q(x, y)|$$

where Q ranges over all indefinite binary quadratic forms.

The Markoff spectrum exhibits surprising and interesting properties; there is no value in the spectrum between $\frac{1}{\sqrt{8}}$ and $\frac{1}{\sqrt{5}}$, and a similar gap between $\frac{1}{\sqrt{13}}$ and $\frac{1}{\sqrt{12}}$. Points like $\frac{1}{\sqrt{5}}$ are of special interest, because in a sense they are only achieved once by all binary quadratic forms [1]. On the other hand, there are accumulation points in the spectrum where there are infinitely many values near the point. The exotic nature of the spectrum makes it an interesting topic. It is often approached from the view of continued fractions, but defining the spectrum through indefinite binary quadratic forms and the use of John Conway's topograph (see [2]), which we construct shortly, allows us an elementary method to see the spectrum.

2 The Topograph

The topograph, first defined by John Conway, of a binary quadratic form is a graphical visualization of the form which allows for simple proofs of many facts about such forms, and will serve as the basis for our approach to visualizing the Markoff spectrum. We develop the topograph based on the exposition in the forthcoming book *An Illustrated Theory of Numbers* by Martin Weissman [3].

2.1 The Domain Topograph

We begin with a visualization of the domain of a form, the **domain topograph**. We begin with the definition of a **basis** of $\mathbb{Z}^2$: $\{(a,b),(c,d)\} \subset \mathbb{Z}^2$ is a **basis** if

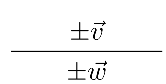
$$\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc = \pm 1.$$

An equivalent definition of a basis $\{\vec{u},\vec{v}\} \subset \mathbb{Z}^2$ is that for any $\vec{w} \in \mathbb{Z}^2$, there are integers m, n such that $m\vec{u} + n\vec{v} = \vec{w}$. If, given any such $\vec{w} \in \mathbb{Z}^2$ we can find integers m, n satisfying this, we say that $\{\vec{u},\vec{v}\}$ **generate** $\mathbb{Z}^2$.

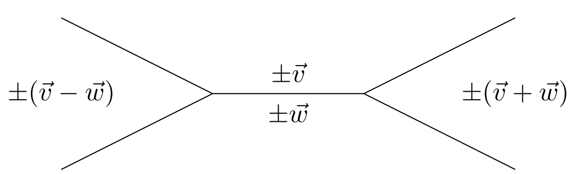
Proposition 1. *A sequence $S = \{(a,b),(c,d)\} \subset \mathbb{Z}^2$ is a basis if and only if the sequence generates $\mathbb{Z}^2$.*

This is a standard result and we omit the proof. We also have that (a,b) participates in a basis if and only if $\gcd(a,b) = 1$, which follows from the fact that the Diophantine equation $ax + by = 1$ has solutions iff $\gcd(a,b) = 1$; any solution will, together with (a,b), form a basis. We are ready now to construct the domain topograph. Additionally, if $\{\vec{u},\vec{v}\}$ is a basis, then $\{\vec{u},\vec{u} \pm \vec{w}\}$ is also a basis. This follows immediately from the fact that if a matrix is obtained from another by adding a constant times one row to another row, then the two matrices have the same determinant. Define a **lax vector** to be any choice of sign for $\pm\vec{v}$ with $\vec{v} \in \mathbb{Z}^2$, and a **lax basis** to be any choice of sign for the elements of $\{\pm\vec{v},\pm\vec{u}\}$. We are now ready to construct the **domain topograph**, a visualization of all lax bases and their relations.

A lax basis $\{\pm\vec{v},\pm\vec{w}\}$ is visualized by a line separating the two vectors:



This should be interpreted as $\{\vec{v},\vec{w}\}$ is a basis for $\mathbb{Z}^2$. We know that $\{\vec{v},\vec{w}\pm\vec{v}\}$ is also a basis, along with $\{\vec{v} \pm \vec{w},\vec{w}\}$; this is represented in the topograph by extending lines from the one separating $\vec{v}$ and $\vec{w}$:



Continuing in this fashion, by starting at the **home basis** $\{\pm(0,1), \pm(1,0)\}$ and adding or subtracting basis vectors, we can reach any line representing a given basis for $\mathbb{Z}^2$. And by using the Euclidean algorithm, we can "walk" from any basis $\{\vec{v}, \vec{w}\}$ to the home basis; if $\vec{v} = q\vec{w} + \vec{r}$ then we can walk from the basis $\{\vec{v}, \vec{w}\}$ to $\{\vec{w}, \vec{r}\}$. So then by carrying out the two dimensional euclidean algorithm, we can walk to the basis $\{(1,u), (0,1)\}$, and from this we can easily walk to $\{(1,0), (0,1)\}$. This demonstrates that all lax bases are connected in the domain topograph.

Some notable structures in the topograph are **triads**, which represents three lax vectors, any two of which form a lax basis:

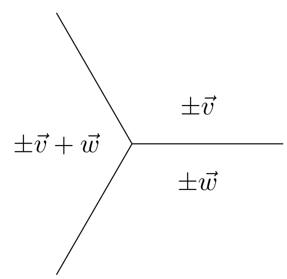


Figure 1: A triad representing the lax vectors $\vec{v}, \vec{w}, \vec{v} + \vec{w}$; any two of these form a lax basis.

Another important structure is a **cell**, a representation of four lax vectors and the five lax bases they make up:

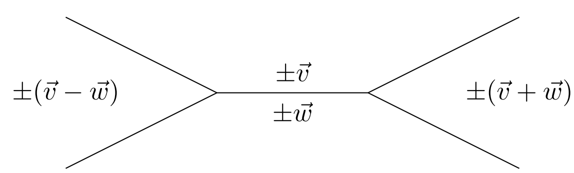


Figure 2: A cell containing four adjacent lax vectors representing five lax bases.

A cell is made up of two triads joined by a common segment.

Finally, if we focus on a particular node, or lax vector, we see an **infinitygon**:

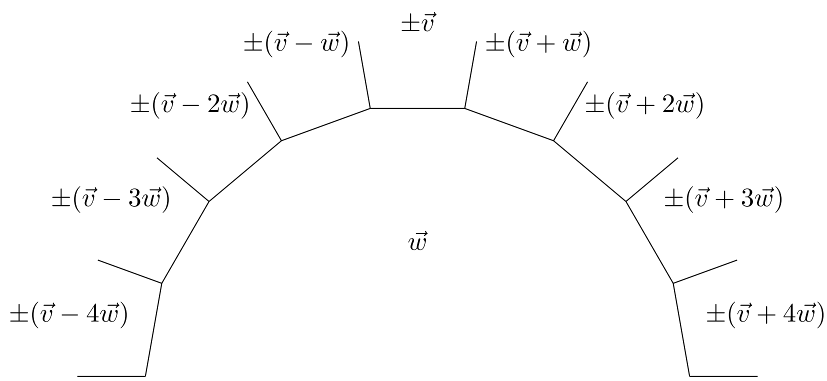


Figure 3: The infinity-gon centered at $\vec{w}$. This structure is infinite because for every integer n, the region containing $\pm(\vec{v} \pm n\vec{w})$ is adjacent to $\pm\vec{w}$.

2.2 The Range Topograph

Once we construct the domain topograph, we can construct the **range topograph** of a binary quadratic form $Q(x, y)$; this is obtained by evaluation of Q at each lax vector (note that $Q(\vec{v}) = Q(-\vec{v})$ and so Q can be unambiguously evaluated at a lax vector). We then plot these values.

2.2.1 Equivalent Binary Quadratic Forms

Before we begin this endeavor, however, we introduce a few standard results about binary quadratic forms. Given two binary quadratic forms Q_1 and Q, we might wonder whether it is always the case that these two forms are different, in that they represent different numbers, or for our purposes, have unique range topographs. This is not the case.

For example, the forms $G(x, y) = ax^2 - bxy + cy^2$ and $F(x, y) = ax^2 + bxy + cy^2$ represent the same integers: if $G(x, y) = n$, then $F(-y, x) = G(x, y) = n$. Say that two forms Q_1 and Q_2 are **equivalent**, $Q_1 \sim Q_2$, if there is an invertible change of coordinates that maps one to the other. More precisely, say that $Q_1 \sim Q_2$ if there is a matrix $M \in GL_2(\mathbb{Z})$ (an integral matrix of determinant ± 1) such that $Q_1(x, y) = Q_2(M(x, y)^T)$. In our example, the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

has determinant one and we have that $G(x, y) = F(M(x, y)^\perp)$.

If $Q(x, y) = ax^2 + bxy + cy^2$, then we can represent Q by a matrix:

$$Q(x, y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Let A be this associated matrix; we have if $M \in GL_2(\mathbb{Z})$ such that M witnesses $Q \sim Q_1$, then if B is the matrix associated with Q_1,

$$B = M^{\perp} A M$$

If Δ is the discriminant of Q, then $\Delta = -4\det(A)$. This shows us that the discriminant is an invariant of equivalent forms: if $Q_1 \sim Q_2$, which have associated matrices A_1 and A_2, and $M \in GL_2(\mathbb{Z})$, then

$$\begin{aligned}\Delta(Q_1) &= -4\det(A_1) = -4\det(M^{\perp} A_2 M) \\ &= -4\det(M^{\perp})\det(A_2)\det(M) \\ &= -4\det(A_2) \\ &= \Delta(Q_2).\end{aligned}$$

2.2.2 Constructing the Range Topograph

We now continue on with the range topograph. The range topograph is formed by evaluating a binary quadratic form at every lax vector in the domain topograph. As mentioned earlier, we can understand the range of a binary quadratic form Q by understanding the image of $\{(a, b)|\, \gcd(a, b) = 1\}$ under Q. Because this is precisely what the range topograph is, it will give us a full understanding of Q.

Our first important feature of the range topopgraph is that we can construct the entire topograph knowning only the values in one cell.

Proposition 2. *Suppose* $\vec{x} = (s, t), \vec{y} = (u, v) \in \mathbb{Z}^2$. *Then*

$$\{Q(\vec{x} - \vec{y}), Q(\vec{x}) + Q(\vec{y}), Q(\vec{x} + \vec{y})\}$$

form an ***arithmetic progression****, a set of the form*

$$\{p + hq | h = 01, 2, \dots, n\}$$

for some fixed $p, q, n \in \mathbb{Z}$.

The proposition follows from an explicit computation, which we omit. When $\vec{v}, \vec{w}$ participate in a lax basis, then the values $Q(\vec{u} - \vec{w}), Q(\vec{u}), Q(\vec{w}), Q(\vec{u} + \vec{w})$ occupy a cell in the range topograph:

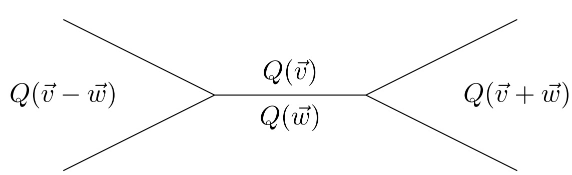


Figure 4: A cell in the range topograph centered at the lax basis $\{\vec{v}, \vec{w}\}$

By using the arithmetic progression property, we can now construct any other part of the range topograph. We just need to know the values of $Q(x, y) = ax^2 + bxy + cy^2$ at a triad to construct the adjacent regions, and thus any region.

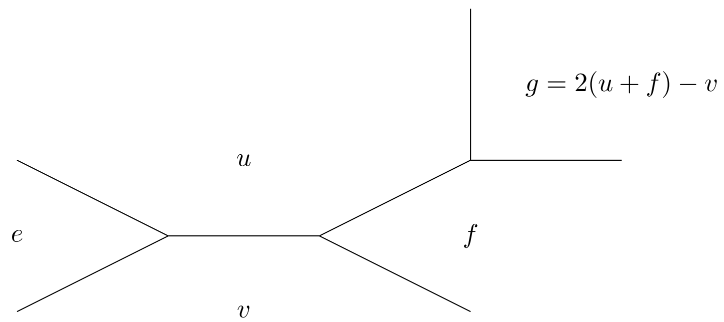


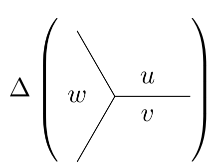
Figure 5: We can calculate the value of g using the arithmetic progression property and the values in the cell containing e, u, v, f.

A consequence of the arithmetic progression property that is central in our search for minima is the *climbing principle*:

Proposition 3 (Climbing Principle)**.** *Suppose e, u, v, f occupy a cell in the range topograph of a binary quadratic form Q, and let $g = 2(u + f) - v$ so that v, u, f, g form a cell as well. If $u > 0$, and $e, u + v, f$ is an increasing arithmetic progression, then the arithmetic progression $v, u + f, g$ is also increasing.*

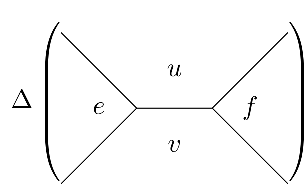
This says that when all values are positive, if they start growing as we develop more of the topograph, then they keep growing.

Another useful property of the range topograph is that we can calculate the discriminant of its associated form using only the values in the topograph. Define a **triad discriminant** of a triad in the range topograph containing w, u, v to be



$$= u^2 + v^2 + w^2 - 2uv - 2vw - 2uw$$

Define a **cell discriminant** containing e, u, v, f to be



$$= (u - v)^2 - ef.$$

Note that these values are independent from our perspective: reflection and rotation of a triad, and reflections of a cell preserve the discriminant. We will see now that, in fact, the discriminant of the range topograph is the same wherever we calculate it.

We can view any two distinct cells in the range topograph as representations of equivalent binary quadratic forms. If $Q(x,y) = ax^2 + bxy + cy^2$, then at the home basis the cell contains $a-b+c, a, c, a+b+c$. If another cell in the range topograph is centered at a lax basis $\{\pm(q,r), \pm(s,t)\}$ then we have an induced form $Q'(x,y) = Q(qx+ry, sx+ty)$. Because this is a basis, $qt - rs = \pm 1$ and thus $Q' \sim Q$. At the home basis, the discriminant of the cell is

$$\begin{aligned}(a-c)^2 - (a+b+c)(a-b+c) &= (a-c)^2 - ((a-b)^2 - c^2)\\ &= -4ac + b^2\\ &= \Delta(Q)\end{aligned}$$

and because the discriminant of any cell is the same as the discriminant of the home basis cell, we have that the discriminants of the cells in the range topograph are all equal. Now if we suppose that, in a given cell with values e, u, v, f, using that these are in arithmetic progression and the invariance under reflection that the discriminant of either triad in the cell is equal to the discriminant of the cell. Thus we may speak of the discriminant of the range topograph, which is also the discriminant of the associated binary quadratic form.

We now begin to study properties of the range topograph of an indefinite binary quadratic form whose discriminant is not a perfect square.

Proposition 4. *Let $Q(x,y) = ax^2 + bxy + cy^2$ be an indefinite binary quadratic form with discriminant Δ. Then Q attains both positive and negative values.*

Proof. Let $g = \gcd(2a, b)$, so that $\gcd(-2a/g, b/g) = 1$. Then we have

$$Q(\frac{b}{g}, \frac{-2a}{g}) = \frac{-a}{g^2}\Delta$$

and we also have that $Q(1,0) = a$. Then because $g^2 > 0$ and we assumed $\Delta > 0$, $Q(b/g, -2a/g)$ and $Q(1,0)$ have opposite signs. $\square$

We now investigate the distribution of these positive and negative values in the domain topograph of an indefinite form. Define a **river** to be a set of line segments in the range topograph which separate positive and negative values, and a **lake** to be a region in which the value is zero. Since we are after the minima of indefinite forms, we wish to exclude those that have a lake in their domain topograph. This leads us to

Proposition 5. *If $Q(x,y) = ax^2 + bxy + cy^2$ is a binary quadratic form with discriminant $\Delta > 0$, then if Δ is not a perfect square, $Q(x,y) \neq 0$ for coprime x, y. That is, there are no lakes in the range topograph of Q.*

Proof. Suppose the range topograph of Q has a lake. Then in a triad around the lake with values $0, f, v$, we can compute the discriminant to be $\Delta = f^2 + v^2 - 2fv = (f-v)^2$, proving the contrapositive. $\square$

So far, if Q is a binary quadratic form with discriminant $\Delta > 0$ which is not a perfect square, we know that the range topograph contains both a positive and negative value, and that it does not have any lakes. We now show

Proposition 6. *There is a river in the range topograph of an indefinite binary quadratic form whose discriminant is not a perfect square.*

Proof. We know the range topograph contains both a positive and negative value. These two regions are connected: there is a path connecting the region with the positive value and the negative value. If every segment connected to this path is not a river, then we must have that every region on either side of the path is negative, since the path connects to a negative value. The path also connects to a positive value, and again because there is no river along the path, every value must be positive. This is impossible, and we reach a contradiction. There must be a river in between the two positive and negative values, and thus the topograph contains a river. □

If Δ is not a perfect square and thus there are no lakes in the topograph, the river cannot end. At a triad containing u, v, f, with a river between u and v, because $f \neq 0$, the river must either fork up or down. Collecting the results so far, we have

Theorem 1. *Suppose $Q(x, y) = ax^2+bxy+cy^2$ is an indefinite binary quadratic form whose discriminant Δ is not a perfect square. Then the range topograph of Q contains an infinite river and no lakes.*

Another result that will prove both useful in understanding the nature of the river, and later in our attempt to see the Markoff spectrum, deals with the growth of values around some fixed value in the range topograph. A **quadratic progression** is a sequence of the form $\{un^2 + hn + t | n = 0, 1, 2, \ldots, k\}$ with $u, h, t \in \mathbb{Z}$ fixed.

Proposition 7. *In the range topograph around a value u, the values form a quadratic progression.*

Proof. Suppose e, u, v, f, g are values in an extended cell in the range topograph. We wish to show that v, f, g and so on take part in a quadratic progression. To prove this, we will construct a polynomial that witnesses this progression. We can think of the cell containing v as the 0th value, f as the first, g as the second, etc, so we seek a quadratic $p(n)$ such that $p(0) = v$, $p(1) = f$, $p(2) = g$, and so on. First, note that due to the arithmetic progression property, $g = 2(u + f) - v$. We claim that $p(n) = un^2 + (f - (u + v))n + v$ witnesses the quadratic progression we seek. As a sanity check, note that indeed $p(0) = v$, $p(1) = u+(f-(u+v))+v = f$, $p(2) = 4u+2(f-(u+v))+v = 2(u+f)-v = g$.

First, we show that for arbitrary k, $p(k - 1), u + p(k), p(k + 1)$ form an arithmetic progression. This will hold if we show that

$$p(k + 1) - (u + p(k)) = u + p(k) - p(k - 1)$$

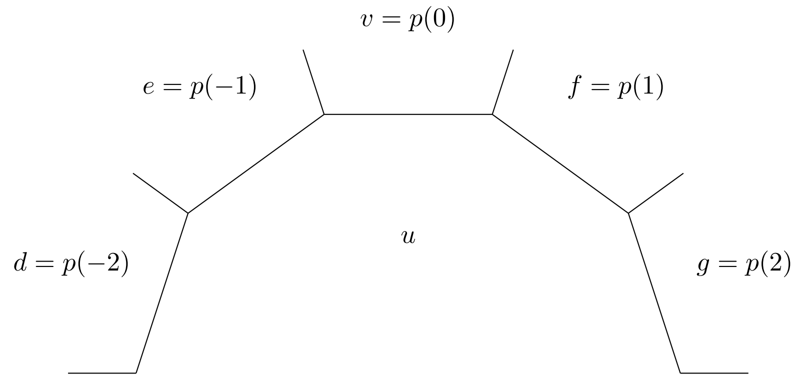


Figure 6: The infinity-gon centered at $\vec{w}$. This structure is infinite because for every integer n, the region containing $\pm(\vec{v} \pm n\vec{w})$ is adjacent to $\pm\vec{w}$.

We compute both sides of this equality.

$$\begin{aligned} p(k+1) - (u + p(k)) &= u(k+1)^2 + (f - (u+v))(k+1) + v \\ &\quad - (u + uk^2 + (f - (u+v))k + v) \\ &= 2uk + f - (u+v) \end{aligned}$$

$$\begin{aligned} u + p(k) - (p(k-1)) &= u + uk^2 + (f - (u+v))k + v \\ &\quad - (u(k-1)^2 + (f - (u+v))(k-1) + v) \\ &= 2uk + f - (u+v) \end{aligned}$$

We conclude that $p(k-1), u + p(k), p(k)$ form an arithmetic progression. Now suppose that, counting from the region containing v, the ith region contains $p(i)$ for $i \leq k$ for k arbitrary. We must show that the $k+1$st region contains $p(k+1)$. But we know that if a cell contains $p(k-1), u, p(k), t$, then since $p(k-1), u + p(k), t$ form an arithmetic progression, $t = p(k+1)$ as desired.

□

A feature of particular interest to our purposes is a **river bend**: a cell containing u, v, e, f where precisely one of u, v and e, f are negative, and the other is positive.

Proposition 8. *The range topograph of an indefinite quadratic form must contain a riverbend*

Proof. This follows from the fact that a river can run along only finitely many sides of an infinity-gon in the topograph; if the infinity-gon is around a value

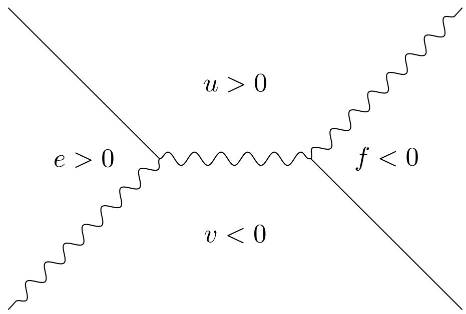


Figure 7: A river bend in the topograph.

u, and without loss of generality, if $u > 0$, then the arithmetic progression is negative and accelerating upwards. Eventually a value must be positive; here the river will bend. $\square$

3 The Markoff Spectrum

We are now prepared to investigate the minima of an indefinite binary quadratic form. Recall the definition of the Markoff spectrum: If $Q(x, y)$ is an indefinite binary quadratic form whose discriminant is not a perfect square, define $m(Q)$ to be

$$m(Q) = \min_{\substack{x,y \in \mathbb{Z} \\ \gcd(x,y)=1}} |Q(x, y)|.$$

We have $m(|t|Q) = |t|m(Q)$ and that $\Delta(tQ) = t^2\Delta(Q)$; thus if one form is obtained by scaling another by some integer, then the quantity

$$\mu(Q) = \frac{m(Q)}{\sqrt{\Delta(Q)}}$$

is equal for both forms; Δ is the proper normalizing factor and $\mu(Q)$ will be the content of the **Markoff Spectrum**, M:

$$M := \{\mu(Q) : Q \text{ is a BQF}, \Delta(Q) > 0, \text{ and } \Delta(Q) \neq k^2\}$$

To see the spectrum, we need to see the minimum value of a given indefinite form. Having uncovered so much order and symmetry in the range topograph, we should not expect the minima of a form Q to be haphazardly strewn about.

Proposition 9. *If $Q(x, y)$ is an indefinite binary quadratic form with discriminant Δ not a perfect square, then $m(Q)$ occurs along the river in the range topograph of Q.*

Proof. Let $a, b \in \mathbb{Z}$ such that $|Q(a,b)| = n$ does not lie on the river. Without loss of generality, assume $Q(a,b) > 0$. If we assume $Q(a,b)$ does not lie on the river, then we may connect it to a path leading to the river. By the climbing principle, as we move away from the river towards $Q(a,b)$, the values increase. Thus any candidate for the minimum value lies along the river. By the well-ordering of the natural numbers, the magnitude of one of the values along the river is minimal. That is, the minimum exists, and it occurs along the river. $\square$

But where along the river shall we look for the minimum value? Recalling that around a value u in the topograph, the adjacent values form a quadratic progression, we know that these adjacent values will change sign at some point. This sign-change occurs at a river bend. The values adjacent to u decrease in one direction until the river bends away from u, and so the values in this cell will be of magnitude less than or equal to that of those away from the river bend. This yields the following:

Theorem 2. *If $Q(x,y)$ is an indefinite binary quadratic form whose discriminant is not a perfect square, then $m(Q)$ occurs at a riverbend in the topograph.*

So in our search for minima, we need only consider values at river bends. We know that given Q an indefinite binary quadratic form, we can find a change of basis such that $Q \sim Q_1$ where Q_1 is a binary quadratic form centered at a river bend. Call such a form $Q(x,y) = ax^2 + bxy + cy^2$ **river-reduced** if its topograph is centered a river-bend. In particular, precisely one of each $\{a, c\}$ and $\{a+b+c, a-b+c\}$ is positive with the other negative. If $b > 0$, we have that this agrees with the usual notion of a **reduced form**: Say that $Q(x,y) = ax^2 + bxy + cy^2$ with discriminant $\Delta > 0$ is **reduced** if

$$\frac{|\sqrt{\Delta} - b|}{2|a|} < 1 < \frac{|\sqrt{\Delta} + b|}{2|a|}.$$

Proposition 10. *If $Q(x,y) = ax^2+bxy+cy^2$ is an indefinite BQF with discriminant Δ not a perfect square, then Q is reduced if and only if Q is river-reduced and $b > 0$.*

The following lemma gives another, similar characterization:

Lemma 1. *An indefinite form $Q(x,y) = ax^2 + bxy + cy^2$ is reduced if and only if $0 < b < \sqrt{\Delta}$ and $0 < \sqrt{\Delta} - b < 2|a| < \sqrt{\Delta} + b$.*

It is also straightforward to see that $ax^2 + bxy + cy^2$ is reduced if and only if $cx^2 + bxy + cy^2$. We are now ready to connect the notion of a reduced form with the properties of the range topopgraph.

Theorem 3. *An indefinite form $Q(x,y) = ax^2 + bxy + cy^2$ with discriminant Δ is reduced if and only if it is river-reduced and $b > 0$.*

Proof. Assume $Q(x, y)$ is reduced. First we show that precisely one of a or c is positive. We have that

$$\begin{aligned} b &< \sqrt{\Delta} \\ b^2 &< b^2 - 4ac \\ 0 &< -4ac \end{aligned}$$

and thus ac is negative, which implies that one of the two must be negative and the other positive. We may assume that $a > 0$. Because $2a = 2|a| < \sqrt{\Delta} + b$ we have

$$\begin{aligned} 2a - b &< \sqrt{\Delta} \\ 4a^2 - 4ab + b^2 &< b^2 - 4ac \\ 4a^2 - 4ab &< -4ac \\ a - b &< -c \end{aligned}$$

and we have that $a - b + c < 0$ as desired. Entirely similarly we get that $a + b + c > 0$. Thus Q is river-reduced.

If Q is river-reduced, then $-4ac > 0$, giving us $b^2 - 4ac > b^2$ and thus $\sqrt{\Delta} > |b| = b$ since $b > 0$. Using that $|a| - b < -c$ we can conclude, reversing the argument above, that $2|a| - b < \sqrt{\Delta}$ and again a similar computation yields $\sqrt{\Delta} - b < 2|a|$ and thus Q is reduced. □

We can also come up with an upper bound on $\mu(Q)$.

Theorem 4. *Suppose e, u, v, f are values in a cell in the range topograph of an indefinite binary quadratic form Q with discriminant Δ not a perfect square, and that this cell contains a river bend. Then we have*

$$|u| \leq \frac{\Delta}{\sqrt{5}} \text{ or } |v| \leq \frac{\Delta}{\sqrt{5}} \text{ or } |e| \leq \frac{\Delta}{\sqrt{5}} \text{ or } |f| \leq \frac{\Delta}{\sqrt{5}}.$$

Proof. Since u and v have opposite signs, e and f have opposite signs, we have that

$$\begin{aligned} \Delta &= (u - v)^2 - ef \\ &= u^2 - uv - vu + v^2 - ef \\ &= |u|^2 + |u| \cdot |v| + |v| \cdot |u| + |v|^2 + |e| \cdot |f| \end{aligned}$$

and thus it cannot be the case that each term in the right hand side is greater than $\Delta/5$. So, for instance, we have $|u| \cdot |v| \leq \Delta/5$ and thus either $|u| \leq \sqrt{\Delta/5}$ or $|v| \leq \sqrt{\Delta/5}$. The other cases are similar. □

This immediately yields

Corollary 1. *For an indefinite binary quadratic form Q, $\mu(Q) \leq \frac{1}{\sqrt{5}}$.*

This bound is sharp but just barely. There is a unique family of reduced binary quadratic form which attains this bound, $\{Q(x,y) = n(x^2+xy-y^2)\}_{n\in\mathbb{Z}}$.

The more important result for our purposes is the following:

Corollary 2. *For a given indefinite binary quadratic form $Q(x,y) = ax^2 + bxy + cy^2$, there are finitely many reduced forms equivalent to Q. That is, the river in the topograph for Q is periodic.*

Proof. At any given cell with a river bend containing u, v, e, f, there is a form $Q'(x,y) = ux^2 + (f-u-v)xy + vy^2$ with this cell as its standard basis cell. From our earlier calculation we saw that we must have $|u| \le \sqrt{\Delta}$, $|v| \le \sqrt{\Delta}$ and since a river reduced form is reduced if $b > 0$, $|v-u-f| \le \sqrt{\Delta}$. So we have finitely many choices for u and v, which then determine f. Thus there are only finitely many distinct river bends in the topograph, and because the river is infinite we have infinitely many river bends, the river must be periodic. Then by the arithmetic progression property, the values along the river everywhere are periodic. $\square$

4 Seeing the Spectrum

To find $\mu(Q)$ for an indefinite binary quadratic form Q with Δ not a perfect square, we need only consider values in its topograph in cells at a river bend. There are finitely many river bends, so there are finitely many candidates for its minimum which we need to check. From another perspective, we may only consider reduced forms; we know that a given form is equivalent to finitely many reduced forms. This will be our approach and the structure of the program that produces a visualization of the Markoff spectrum: rather than fixing a form and finding its minimum and iterating this process over all indefinite forms, we fix a discriminant and find the minima of all possible reduced forms with that given discriminant.

We need to know how to move from one reduced form to another for a given form, that is, how to step along the river to the next river bend. Suppose $Q(x,y) = ax^2 + bxy + cy^2$ is river-reduced with $b > 0$. Then $ac < 0$; suppose $a < 0$ and $c > 0$. There is then a river bend at the standard basis, where $Q(x,y)$ is centered. If we follow the river to the right, looking for the next river bend, we will see the river move along the infinity-gon surrounding the value c, but we know it must eventually fork away. We seek a way to determine where this occurs, based on the values of a, b, and c.

Suppose that the river in the topograph travels along the sides of an infinity-gon containing c. Recall that the values surrounding c form a quadratic progression. That is, there is a degree two polynomial $p : \mathbb{Z} \to \mathbb{Z}$ such that $p(0), p(1), \ldots, p(n)$ form a quadratic progression which agree with the values surrounding c. We have $p(0) = a$, $p(1) = a + b + c$, $p(2) = a + 2b + 4c$, and so on. Thus $p(n) = cn^2 + bn + a$. To find where the river bends again, we simply need to find a k such that $p(k), p(k+1)$ have opposite signs. The roots of p are given by

$$n = \frac{-b \pm \sqrt{\Delta}}{2c}$$

and thus if x_0 is the positive root, $\lfloor p(x_0) \rfloor$ and $\lceil p(x_0) \rceil$ give the region "above" c and to the right of c at the next river bend. We can then determine the value in the region to the left of c based on this information. Note that roots of $p(n)$ as defined above are given by

$$x_0 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2c}$$

and since $\Delta(Q) = b^2 - 4ac > 0$ since we assume Q to be indefinite, these roots are real and distinct, and one is positive. We choose x_0 to be this positive root.

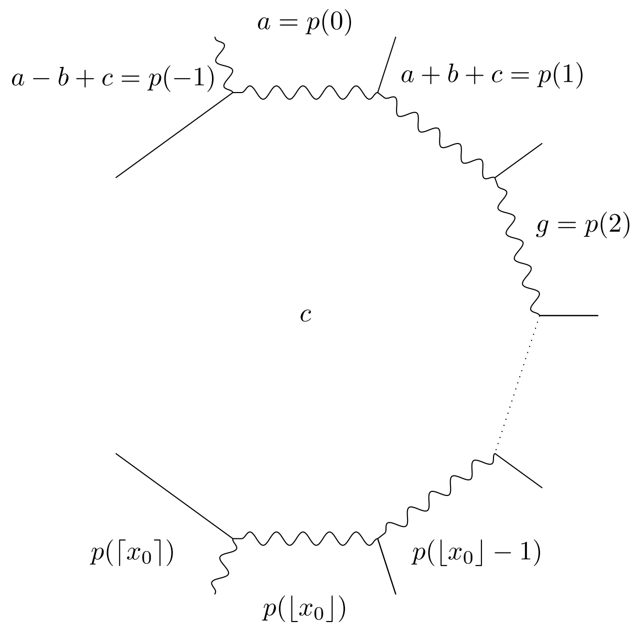


Figure 8: The river traveling around the infinity-gon containing u. The river bends when $p(n)$ changes sign; if x_0 is the positive root of $p(n)$, this sign change occurs at $p(\lfloor x_0 \rfloor)$ and $p(\lceil x_0 \rceil)$.

This gives us a method for moving from one river bend to another. If $Q(x, y) = ax^2 + bxy + cy^2$ is river reduced, say, with $a > 0$, $b > 0$, $c < 0$ and has discriminant Δ, then $p(n) = cn^2 + bn + a$. Its positive root is given by

$$x_0 = \frac{b + \sqrt{\Delta}}{-2c}$$

which we can calculate by finding $m \in \mathbb{N}$ such that

$$m(-2c) \leq b + \sqrt{\Delta} \leq (m + 1)(-2c)).$$

We can simplify this to the task of performing integer division of $b + \left\lfloor \sqrt{\Delta} \right\rfloor$ by $(-2c)$ and discarding the remainder. Once this root is found, then, the new river bend cell is obtained and we can find $Q'(x, y) \sim Q(x, y)$ such that

$$Q'(x, y) = p(\lfloor x_0 \rfloor)x^2 + (p(\lceil x_0 \rceil) - p(\lfloor x_0 \rfloor) - c)xy + cy^2$$

is also reduced.

Now the simple task of finding the minimum value of Q is to step along the river, from bend to bend, and collect the minimum of the four regions in the cell. The smallest such of all these will be the minimum value of Q.

To plot the Markoff Spectrum (of forms less than a given discriminant) we first need a way to generate reduced quadratic forms. First, note

$$\Delta = b^2 - 4ac \equiv 0 \text{ or } 1 \mod 4.$$

If Q is to be reduced, we know that $0 < b < \sqrt{\Delta}$. If $\Delta \equiv 0 \mod 4$, then b is even; $b = 2\beta$ for $\beta = 1, 2, \dots$ and $\beta < \sqrt{\Delta}/2$. Once we determine β and thus b, if we set

$$e = \Delta - b^2 = -4ac,$$

by choosing a to be any divisor of $e/4$, we also have $c = -e/(4a)$ determined. Having done this, we need only pass a, b, c satisfying $a+b+c > 0$ and $a-b+c < 0$ to the river stepping algorithm, and then find the minimum for all forms equivalent to that given Q. The case for $\Delta \equiv 1 \mod 4$ is similar: now $b = 2\beta+1$ for some $\beta < (\Delta - 1)/2$ and a, c are determined in the exact same way.

We can also find the second smallest output of a binary quadratic form this way. By the climbing principle, we know that the magnitude of values in the regions increases as we travel further from the river, and we know that they are smallest at the river bend. This means we need to only check values at the river bend and at adjacent cells[1] to find the second smallest output.

[1] It is necessary to look away from the river to find a true second minimum for some forms. In particular, the form $Q(x, y) = x^2 + xy - y^2$ only has values of ± 1 along its river, and to see its second minimum of 5 we must look at adjacent cells.

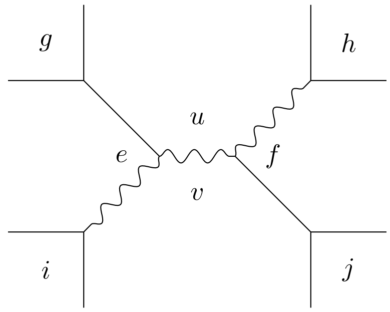


Figure 9: Candidates for the second minimum of a BQF.

The actual implementation of this code in sage is included in 4.1. Without further ado, the output of that code: the Markoff spectrum.

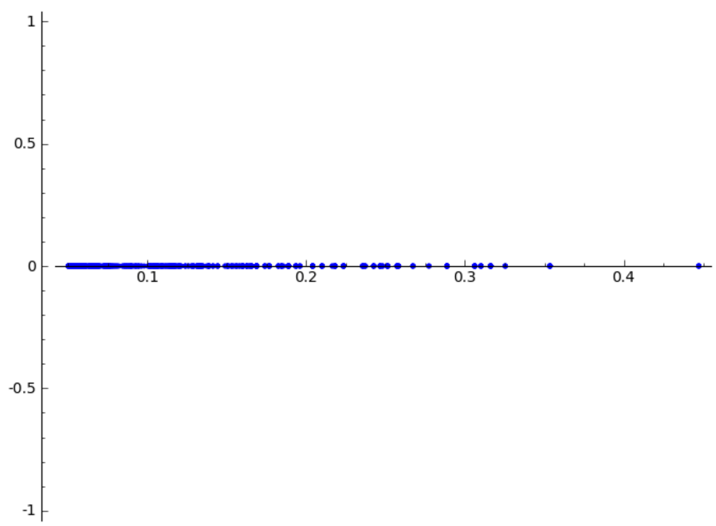


Figure 10: The Markoff Spectrum for $\Delta(Q) < 400$.

We see indeed, as promised, there is a lonely point at $\frac{1}{\sqrt{5}} \approx 0.447213595$, and another solitary minimum at $\frac{1}{\sqrt{8}} \approx 0.353553391$. Nearby is $\frac{1}{\sqrt{12}}$ followed closely by $\frac{1}{\sqrt{13}}$. As these values decrease, we see their density growing. As the discriminant increases we find more and more accumulation points.

As mentioned, the program also computes a second smallest normalized value. We plot these against the corresponding value in the Markoff spectrum:

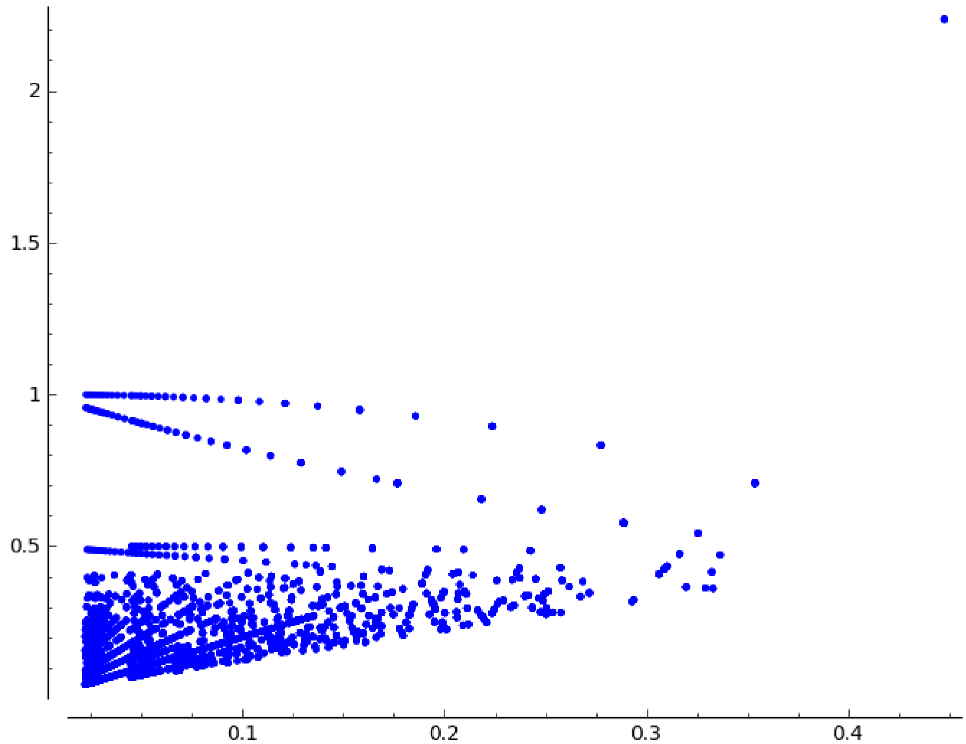


Figure 11: Second normalized minima vs first normalized minima for $\delta(Q) < 4000$.

This plot reveals further interesting behavior of these indefinite forms. Note the several sequences of normalized values that seem to converge to some limit; there is a prominent sequence appearing to converge to one as $\Delta \to \infty$. What this may correspond to is when the second smallest minimum value of $Q(x, y) = ax^2 + bxy + cy^2$ occurs in the region in the topograph occupied by $a + b + c$. What might be happening is that $a = c$ and $a, c \ll b$ so that $\Delta \approx b^2$. Then we have that

$$\frac{a+b+c}{\sqrt{\Delta}} \approx \frac{b}{\sqrt{b^2}} \approx 1$$

and this is more likely to occur as Δ grows larger. The sequence approaching $1/2$ might be due to cases in which b is significantly smaller than Δ, $a \approx c$ and the minimum is a, second minimum c so that the normalized second minimum is $\frac{c}{\sqrt{\Delta}} \approx \frac{c}{\sqrt{4c^2}} \approx 1/2$. But these are merely guesses.

4.1 The Code

The following is a program written in python and executable in sage which will compute simultaneously both the normalized minimum and normalized second minimum of all binary quadratic forms having discriminant less than $4d$ where d is the input into MarSpec().

```
#Finds the two smallest values of a sequence.
def FindTwoMin(x):
    m=min(x)
    #Remove all instances of the minimum to find the second smallest.
    while m in x:
        x.remove(m)
    #If the sequence is constant, take the second minimum to be the first.
    if x==[]:
        n=m
    else:
        n=min(x)
    return m,n

#Returns the next coefficients of a BQF whose
#home basis cell is at a river bend
def RiverStep(a,b,c):
    Delta=b*b-4*a*c
    IRootDelta=isqrt(Delta)
    #The values around c form a quadratic progression given by cn^2+bn+a
    #The progression changes sign at a positive root
    #The floor and ceiling of the root give the positive and negative
    #Values along the river, respectively
    #Finally determine 'new' a,b,c such that ax^2+bxy+cy^2 has as its home
    #basis cell the next river bend.
    if (a>0 and a+b+c>0):
        #This is the floor of the positive root
        NextRoot=(b+IRootDelta)//(-2*c)
        NextRootPlusOne=NextRoot+1
        Nextabc = c*(NextRootPlusOne)^2 + b*(NextRootPlusOne) + a
        Nexta=c*(NextRoot)^2 + b*(NextRoot) + a
        Nextc=c
        Nextb=Nextabc-Nexta-Nextc
    #Here, the values along the river are given by an^2+bn+c
    if (a>0 and a+b+c<0):
        #This is the floor of the positive root
        NextRoot=(-b+IRootDelta)//(2*a)
        NextRootPlusOne=NextRoot+1
        Nextabc = a*(NextRootPlusOne)^2 + b*(NextRootPlusOne) + c
        Nextc=a*(NextRoot)^2 + b*(NextRoot) + c
```

```
            Nexta=a
            Nextb=Nextabc-Nexta-Nextc
        #The values along river are given by cn^2+bn+a
        if (a<0 and a+b+c<0):
            #This is the floor of the positive root
            NextRoot=(-b+IRootDelta)//(2*c)
            NextRootPlusOne=NextRoot+1
            Nextabc = c*(NextRootPlusOne)^2 + b*(NextRootPlusOne) + a
            Nexta=c*(NextRoot)^2 + b*(NextRoot) + a
            Nextc=c
            Nextb=Nextabc-Nexta-Nextc
        #Values along river are given by an^2+bn+c
        if (a<0 and a+b+c>0):
            #This is the floor of the positive root
            NextRoot=(b+IRootDelta)//(-2*a)
            NextRootPlusOne=NextRoot+1
            Nextabc = a*(NextRootPlusOne)^2 + b*(NextRootPlusOne) + c
            Nextc=a*(NextRoot)^2 + b*(NextRoot) + c
            Nexta=a
            Nextb=Nextabc-Nexta-Nextc

        return Nexta,Nextb,Nextc



# Finds minimum value of regions in cell at standard basis
# and of those regions adjacent to that cell
def TwoCellMin(a,b,c):
    e=a-b+c
    f=a+b+c
    g=2*(e+a)-c
    h=2*(a+f)-c
    i=2*(e+c)-a
    j=2*(c+f)-a
    L=[abs(a),abs(c),abs(e),abs(f),abs(g),abs(h),abs(i),abs(j)]
    m,n=FindTwoMin(L)
    return m,n

#Given a BQF at a river bend, finds min value of all
#values in regions in cells with river bends for that BQF
def BQFTwoMin(a,b,c):
    M,N=TwoCellMin(a,b,c)
    A,B,C=a,b,c
    a,b,c=RiverStep(a,b,c)
    while (a,b,c)!=(A,B,C):
        #Step to next bend and find first two minimums among
```

```
            #the previous two minimums and values in the cell
            #and adjacent cells.
            a,b,c=RiverStep(a,b,c)
            M,N=FindTwoMin([M,N,TwoCellMin(a,b,c)[0],TwoCellMin(a,b,c)[1]])
        return M,N

# This function produces a sequence of
# ordered pairs for all BQF of discriminant
# less than 4r. The ordered pairs consist of
# [m1(Q)/sqrt(Delta(Q)),m2(Q)/sqrt(Delta(Q))] where
# m1 is the minimum value of Q, m2 is the second minimum.
# This can be plotted in sage with list_plot(MarSpec(r)).
def MarSpec(r):
    #Initiate an empty list
    x=[]
    #Delta is congruent to 0 or 1 mod 4
    for d in range(1,r):
        Delta=4*d
        if not(is_square(Delta)):
            #b is even, b=2Beta
            #we only consider reduced forms, so 0<b<sqrt(Delta)
            for Beta in range(1,1+ (sqrt(Delta)/2)):
                b=2*Beta
                e=Delta-b*b
                #e=-4ac so a must divide e//4
                #RiverStep only works properly for
                #variables stored as integers
                #e is an integer congruent to 0 mod 4 so e/4=e//4
                for a in divisors(e//4):
                    c=-e//(4*a)
                    #Only pass forms to RiverStep which are already
                    #at river bend (thus reduced)
                    if (0<(a+b+c) and 0>(a-b+c)):
                        #Find the two minimum values
                        m,n=BQFTwoMin(a,b,c)
                        #Normalize the first two minimums of the BQF
                        r=m/numerical_approx(sqrt(Delta))
                        s=n/numerical_approx(sqrt(Delta))
                        #Add this normalized ordered pair to x
                        x.append([r,s])

    for d in range(1,r):
        #The case when Delta congruent to 1 mod 4
        #Now b is odd
        #The algorithm is the same as above
        Delta=4*d+1
```

```
    if not(is_square(Delta)):
        for Beta in range(0,(sqrt(Delta)+1)/2):
            b=2*Beta+1
            e=Delta-b*b
            for a in divisors(e//4):
                c=-e//(4*a)
                if (0<(a+b+c) and 0>(a-b+c)):
                    m,n=BQFTwoMin(a,b,c)
                    r=m/numerical_approx(sqrt(Delta))
                    s=n/numerical_approx(sqrt(Delta))
                    x.append([r,s])
return x
```

References

[1] Thomas W. Cusick, Mary E. Flahive, *The Markoff and Lagrange Spectra*. American Mathematical Society, Rhode Island, 1989.

[2] John H. Conway, *The Sensual Quadratic Form*. The Mathematical Association of America, 1997.

[3] Martin Weissman, *An Illustrated Theory of Numbers*. Lecture notes, 2011.